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Mathematics 205A, Fall 2014, Examination 1

Answer Key

1. [*20 points*] Suppose that $(X_1, \mathbf{T}_1)$ and $(X_2, \mathbf{T}_2)$ are topological spaces, and let $\mathbf{U}$ be a topology on $X_1 \times X_2$ such that the two coordinate projections $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$ are continuous. Prove that $\mathbf{U}$ contains the product topology on $X_1 \times X_2$. [*Hint:* Why is it enough to prove that $\mathbf{U}$ contains a base for the product topology?]

SOLUTION

If we are given a base for a topology $\mathbf{P}$, then every subset in $\mathbf{P}$ is a union of sets in $\mathbf{P}$. Since $\mathbf{U}$ is closed under unions, it contains all of $\mathbf{P}$ if it contains a base for $\mathbf{P}$.

Now specialize to the case where $\mathbf{P}$ is the product topology, and take the base consisting of product sets $V_1 \times V_2$ where V_i is open in X_i for $i = 1, 2$. By continuity the sets $p_i^{-1}[V_i]$ are $\mathbf{U}$-open. Since $\mathbf{U}$ is closed under intersections it follows that the intersection of these sets is also in $\mathbf{U}$. But

$$p_1^{-1}[V_1] \;\cap\; p_2^{-1}[V_2] \quad = \quad V_1 \times V_2$$

and therefore the continuity of p_1 and p_2 implies that $V_1 \times V_2$ is in $\mathbf{U}$. By the remarks in the preceding paragraph, it follows that all of $\mathbf{P}$ is contained in $\mathbf{U}$.■

2. [*25 points*] Let X be a compact topological space, let Y be a topological space which satisfies the Hausdorff Separation Property, and let $f : X \to Y$ be a continuous mapping which is 1–1 and onto. Prove that f is a homeomorphism.

SOLUTION

A continuous and 1–1 onto map from one space to another is a homeomorphism if and only if it sends closed subsets to closed subsets. Therefore it is enough to show that f has this property. But $F \subset X$ closed and X compact implies F is compact, which implies that $f[F]$ is compact, which implies that $f[F]$ is a closed subset of the Hausdorff space Y.■

3. [*30 points*] (*a*) Let X be set with the finitary topology **Fin** (a proper subset is closed if and only if it is finite). Under what conditions on X is $(X, \mathbf{Fin})$ connected? [*Caution:* The correct answer involves 2 or 3 mutually exclusive possibilities!]

(*b*) Let $A \subset \mathbb{R}$, let $X \subset \mathbb{R}^2$ be the open upper half plane consisting of all (x, y) such that $y > 0$, and let $C_A = X \cup A \times \{0\}$. Explain why C_A is a connected subset of $\mathbb{R}^2$. [*Hint:* What is the closure of X in $\mathbb{R}^2$?]

(*c*) Let $V \subset \mathbb{R}^2$ be the complement of the x-axis (= all points whose second coordinates are nonzero). Is V connected? Is V locally connected? Give brief explanations for your answers.

SOLUTION

(*a*) The finitary and discrete topologies are equal for finite sets, so the finitary topology on a finite set is connected if and only if the set has at most one point. — If X is infinite, then X must be connected, for if $X = A \cup B$ where A and B are closed subsets, then at least one of them must be infinite. Since X itself is the only closed subset which is infinite, it is clear that A and B cannot be disjoint.■

(*b*) More generally, if $E \subset Y$ is connected and $E \subset B \subset \overline{E}$, then B is connected. — Applying this to the example, the open square X is connected because it is the product of two open intervals, and the closure of X in $\mathbb{R}^2$ is just the solid square $[0,1] \times [0,1]$. Therefore we have $X \subset C_A \subset \overline{X}$, which means that C_A is connected.■

ALTERNATE ARGUMENT. In fact, C_A is arcwise connected. The space X itself is arcwise connected because it is a product of arcwise connected spaces, and every point in $A \times \{0\}$ can be joined to a point in X by a vertical line segment and this segment is contained in C_A.■

(*c*) V is locally connected because it is open in the locally connected space $\mathbb{R}^2$, and it is not connected because the upper and lower half planes, defined by $y > 0$ and $y < 0$ respectively, are open and closed subsets of V.■

4. [*25 points*] (a) Let $(X, \mathbf{T})$ be a topological space, let Y be a set, and let $f : X \to Y$ be an onto function. Define the quotient topology $f_*\mathbf{T}$ on Y determined by $(X, \mathbf{T})$ and f, and explain why the map $f : (X, \mathbf{T}) \to (Y, f_*\mathbf{T})$ is continuous.

(b) Let $(X, \mathbf{T})$, Y and f be as in (a), and suppose that $\mathbf{U}$ is a topology on Y such that $f : (X, \mathbf{T}) \to (Y, \mathbf{U})$ is continuous and open. Prove that $\mathbf{U} = f_*\mathbf{T}$. [*Hint:* Show that a set is $\mathbf{U}$-open if and only if it is $f_*\mathbf{T}$-open. — *Caution:* In general the map $f : (X, \mathbf{T}) \to (Y, f_*\mathbf{T})$ is not necessarily open.]

SOLUTION

(a) The quotient topology on Y consists of all sets $V \subset Y$ such that $f^{-1}[V]$ is open in X.■

(b) Every $\mathbf{U}$-open set is $f_*\mathbf{T}$-open because if V is $\mathbf{U}$-open then $f^{-1}[V]$ is $\mathbf{T}$-open by continuity. Conversely, suppose that $f^{-1}[V]$ is $\mathbf{T}$-open. To see that V is $\mathbf{U}$-open, note that the open mapping condition on f implies that $f\left[f^{-1}[V]\right]$ is $\mathbf{U}$-open. Since f is onto we have $V = f\left[f^{-1}[V]\right]$, and therefore it follows that V is $\mathbf{U}$-open if $f^{-1}[V]$ is $\mathbf{T}$-open.■