Mathematics 205A, Fall 2014, Examination 2

Answer Key

1. [*25 points*] Let $(X, \mathbf{T})$ be a (nonempty) topological space which is second countable. Prove that $(X, \mathbf{T})$ has a countable dense subset.

SOLUTION

Let $\mathcal{B} = \{U_1, U_2, \cdots\}$ be a countable base for the topology; we might as well assume each subset is nonempty. For each positive integer j choose some point $d_j \in U_j$, and let $D = \{d_1, d_2, \cdots\}$. We claim that D is dense in X.

It suffices to show that if $V \subset X$ is a nonempty open subset, then $V \cap D \neq \emptyset$. But if V is a nonempty set, then V is a union of subsets $U_{\alpha(1)}$, $U_{\alpha(2)}$, *etc.* in $\mathcal{B}$. In particular, this means that $d_{\alpha(1)} \in U_{\alpha(1)} \cap D \subset V \cap D$, so the latter is indeed nonempty.■

2. [*25 points*] (*a*) State the definition of a regular topological space.

(*b*) Let $(X, \mathbf{T})$ be a locally compact Hausdorff topological space which is not compact, and let $(X^{\bullet}, \mathbf{T}^{\bullet})$ be its one point compactification. Define the underlying topology $\mathbf{T}^{\bullet}$.

SOLUTION

(*a*) There are two equivalent definitions: (*i*) Given a point x and an open set U such that $x \in U$, then there is some open subset V such that $x \in V \subset \overline{V} \subset U$, where the middle set is the closure in X. (*ii*) Given a point x and a closed set F such that $x \notin F$, there are disjoint open subsets U and V such that $x \in U$ and $F \subset V$. *Either answer is acceptable.*■

(*b*) By construction $X^{\bullet}$ is the disjoint union $X \amalg \{\infty\}$. The open subsets of $X^{\bullet}$ are the open subsets of X together with subsets of the form $(X - K) \amalg \{\infty\}$, where K is a compact subset of X.■

3. [*30 points*] (a) Let $(X, \mathbf{T})$ be a (nonempty) topological space. Prove that $X \times \{0\}$ is a strong deformation retract of $X \times [0,1]$.

(b) Suppose that $(X, \mathbf{T})$ is a nonempty arcwise connected space and we are given homotopic continuous mappings $f, g : (X, \mathbf{T}) \to (Y, \mathbf{U})$. Explain why there is a single arc component $A \subset Y$ which contains both $f[X]$ and $g[X]$.

SOLUTION

(a) Let $\rho : X \times [0,1] \to X$ be the map sending (x,t) to x, and let i_0 denote the inclusion $X \times \{0\} \subset X \times [0,1]$. Then $\rho \circ i_0 = \mathrm{id}_X$ and the straight line homotopy $H(x,t;s) = (x, (1-s)t)$ defines a homotopy from the identity on $X \times [0,1]$ to the map $i_0 \circ \rho(x,t) = (x,0)$. Since $H(x,0;s) = (x,0)$ for all s, this homotopy is fixed on $X \times \{0\}$, and therefore the existence of $i_0 \circ \rho$ and H shows that $X \times \{0\}$ is a strong deformation retract of $X \times [0,1]$.■

(b) Let $H : X \times [0,1]$ be a homotopy from f to g. Since X and $[0,1]$ are arcwise connected, their product is also arcwise connected, and therefore the image of H must be arcwise connected. This image lies in some arc component A of X. Since f and g are given by restricting H to $X \times \{0\}$ and $X \times \{1\}$, it follows that the images of f and g must be contained in A.■

4. [*20 points*] Let n be a positive integer, and define the n-dimensional torus T^n to be the Cartesian product of n copies of the circle S^1. Prove that T^n and T^m are not homeomorphic if $m \neq n$. [*Hint:* What are the fundamental groups of these spaces?]

SOLUTION

If e denotes the usual basepoint of T^k whose coordinaes are all 1's, then $\pi_1(T^k, e) \cong \mathbb{Z}^k$. In particular, the fundamental groups of T^n and T^n are isomorphic to $\mathbb{Z}^n$ and $\mathbb{Z}^m$ respectively. Since these groups are not isomorphic if $m \neq n$, it follows that T^n and T^m are not homeomorphic if $m \neq n$.■