Mathematics 205A, Fall 2014, Examination 3

Answer Key

1. [*15 points*] Suppose that X is a compact topological space and we are given a nestedsequence of nonempty closed subsets $F_1 \supset F_2 \supset \cdots$. Prove that $\cap_{n=1}^{\infty} F_n$ is nonempty.

SOLUTION

Let $U_n = X - F_n$. Then by hypothesis we have $U_1 \subset U_2 \subset \cdots$ and each U_n is a proper subset of X. Now $\cap_{n=1}^{\infty} F_n$ is nonempty if and only if $\cup_{n=1}^{\infty} U_n$ is a proper subset of X, so it suffices to prove the latter assertion.

Suppose that $\cup_{n=1}^{\infty} U_n = X$. Then $\mathcal{U} = \{U_n\}$ is an open covering of X, so by compactness there is a finite subcovering. Let U_N be the last open set in this subcovering. Since $U_1 \subset U_2 \subset \cdots$ it follows that $U_N = X$. But this contradicts the fact that each U_n is a proper subset; the source of this contradiction was the first sentence in this paragraph, and therefore $\cup_{n=1}^{\infty} U_n$ must be a proper subset of X. As noted before, this means that $\cap_{n=1}^{\infty} F_n$ is nonempty.■

2. [*20 points*] If X is a topological space and A is a closed subset of X, let X/A denote the quotient of X by the equivalence relation whose equivalence classes are (i) all one point sets $\{x\}$ such that $x \notin A$ and (ii) the set A, and let $p : X \to X/A$ denote the quotient space projection. Suppose that $N \subset X/A$ contains the equivalence class $[A]$. Prove that N is an open neighborhood of $[A]$ if and only if $p^{-1}[N]$ is an open subset of X containing A.

SOLUTION

The set $N \subset X/A$ contains the equivalence class $[A]$ if and only if $p^{-1}[N]$ contains A, and N is open if and only if $p^{-1}[N]$ is open.■

3. [*25 points*] Let X and Y be locally compact Hausdorff spaces. Prove that the product space $X \times Y$ is also locally compact Hausdorff. [*Hint:* Recall that $\overline{A \times B} = \overline{A} \times \overline{B}$.]

SOLUTION

First, a product of Hausdorff spaces is Hausdorff. If $(x_1, y_1) \neq (x_2, y_2)$, then either $x_1 \neq x_2$ or $y_1 \neq y_2$. In the first case there are disjoint open subsets in X, say U_1 and U_2, such that $x_i \in U_i$ and $U_1 \cap U_2 = \emptyset$. It follows that $U_1 \times Y$ and $U_2 \times Y$ are disjoint open subsets containing the original two points. — In the second case there are disjoint open subsets in Y, say V_1 and V_2, such that $y_i \in V_i$ and $V_1 \cap V_2 = \emptyset$. It follows that $X \times V_1$ and $X \times V_2$ are disjoint open subsets containing the original two points.

It remains to prove that $X \times Y$ is locally compact. Let $(x, y) \in X \times Y$, and let W be an open subset containing (x, y). By the definition of the product topology there are open neighborhoods U_0 and V_0 of x and y such that $(x, y) \in U_0 \times V_0 \subset W$. Furthermore, since X and Y are locally compact Hausdorff there are open subneighborhoods U and V such that $x \in U \subset \overline{U} \subset U_0$ and $y \in V \subset \overline{V} \subset V_0$ where $\overline{U}$ and $\overline{V}$ are both compact. We then have

$$(x, y) \quad \in \quad U \times V \quad \subset \quad \overline{U} \times \overline{V} \quad \subset U_0 \times V_0 \quad \subset W \ .$$

The product of the closures is compact because it is the product of two compact spaces, and as noted in the hint it is equal to the closure of $U \times V$. Therefore we have constructed an open neighborhood W_0 of (x, y) whose closure is compact and contained in W, and hence $X \times Y$ is locally compact.■

4. [*20 points*] Prove that the unit sphere S^n is a strong deformation retract of $\mathbb{R}^{n+1} - \{\mathbf{0}\}$.

SOLUTION

Define the retraction $\rho : \mathbb{R}^{n+1} - \{\mathbf{0}\} \to S^n$ by $\rho(x) = |x|^{-1} \cdot x$; by construction $\rho|S^n$ is the identity. Let $i : S^n \to \mathbb{R}^{n+1} - \{\mathbf{0}\}$ be the inclusion, and define a homotopy from $i \circ \rho$ to the identity by the straight line homotopy.

$$H(x,t) \quad = \quad (1-t)\,x \;+\; t \circ \rho(x)$$

We then have $H_0 =$ identity, $H_1 = i \circ \rho$, and $H(x,t) = x$ for $x \in S^n$ and t arbitrary, so the only thing to check is that $H(x,t)$ is never zero.

Since $\rho(x)$ is a scalar multiple of x, it follows that $H(x,t)$ is always a scalar multiple of x, and hence it is enough to show that the coefficient of x in the formaula is always nonzero. But this coefficient is

$$(1-t) \;+\; t \cdot |x|^{-1}$$

where $|x| > 0$ and $t \in [0,1]$. If $t = 0$ this coefficient reduces to 1, and if $t > 0$ then also $1 - t > 0$, and therefore the coefficient is a positive real number. Therefore $H(x,t)$ is never zero and hence defines a homotopy from $\left(\mathbb{R}^{n+1} - \{\mathbf{0}\}\right) \times [0,1]$ to $\mathbb{R}^{n+1} - \{\mathbf{0}\}$.■

5. [*20 points*] Suppose that the continuous mapping $p : X \to Y$ is a covering space projection and that Y is connected. Prove that if y_0 and y_1 are arbitrary points of Y, then the cardinalities of the sets $p^{-1}[\{y_0\}]$ and $p^{-1}[\{y_1\}]$ are equal. [*Hint:* Why is this true if y_1 lies in an evenly covered open neighborhood of y_0?]

SOLUTION

Define an equivalence relation $\mathcal{E}$ on Y such that two points are $\mathcal{E}$-equivalent if and only if their inverse images in X have the same cardinality. If $y_0 \in Y$ and U is an evenly covered open neighborhood of y_0, the it follows immediately that every point in U is $\mathcal{E}$-equivalent to y_0. Therefore each $\mathcal{E}$-equivalence class is open in Y.

If E is an equivalence class, the reasoning of the preceding paragraph implies that the union of all the remaining equivalence classes is also open, and therefore E is also closed in Y. But Y is connected, so the only nonempty open and closed subset of Y is Y itself. Therefore if E is an equivalence class, then $E = Y$, and this means that for all point the inverse images have the same cardinality.■