Mathematics 205A, Fall 2014, "Examination 4" Problems

Although Sections VIII.5 and Unit IX of `fundgp-notes.pdf` were not covered on the in-class examinations, they are part of the material covered on the Mathematics 205 qualifying examination. This document contains some problems which might have appeared on an in-class examination if there had been adequate time for students to review and digest the material in these parts of the course (possibly these are more challenging than questions which actually might have appeared on an in-class examination, but they certainly could have been selected for a take-home examination). Therefore, studying them and finding the solutions is highly recommended.

1. (i) Let $\mathbb{RP}^k$ denote real projective n-space where $k \geq 1$. Prove that $\mathbb{RP}^1$ is not homeomorphic to a retract of $\mathbb{RP}^n$ if $n \geq 2$. [*Hint:* $\mathbb{RP}^1$ is homeomorphic to S^1.]

(ii) Viewing S^1 as the unit circle in the complex plane, define $f : S^1 \to S^1$ by $f(z) = z^2$. Prove that f is a covering space projection. How many points are in the inverse image of $f^{-1}(z_0)$ for some arbitrary $z_0 \in S^1$?

2. Let $n, k \geq 2$ be positive integers. Porve that there is a group G with the following properties:

(1) G has k generators.

(2) $g^n = 1$ for all $g \in G$.

(3) If H is a group satisfying the two preceding conditions, then H is isomorphic to a quotient group of G. [*Hint:* One can construct G as a quotient of a free group on k generators.]

Note. A major question in group theory, known as the *Burnside Problem*, asks whether groups satisfying the first two conditions must be finite. If $n = 2$ then it is a fairly standard exercise in group theory to show that the group is abelian and that the answer to the problem is yes, but things become complicated very quickly for higher values of n. The *Wikipedia* article

`http://en.wikipedia.org/wiki/Burnside%27s_problem`

contains a concise and fairly clear summary of results on this question.

3. (i) Suppose that the topological space X is a union of the arcwise connected open subspaces U and V where $U \cap V$ is also arcwise connected, and let $p \in U \cap V$. Suppose further that the map of fundamental groups $\pi_1(U \cap V, p) \to \pi_1(V, p)$ corresponds to an inclusion mapping $H \to G$ where H is a normal subgroup of G, and also that V is simply connected. Prove that $\pi_1(X, p)$ is isomorphic to G/H.

(ii) Let $n \geq 3$, and define $p_r \in \mathbb{R}^n$ to be the point whose first coordinate is r and whose remaining coordinates are zero. Prove that $\mathbb{R}^n - \{p_0, p_1\}$ is simply connected. [*Hint:* Write the space in question as $U \cap V$ where $U \subset \mathbb{R}^n - \{p_0, p_1\}$ is all points whose first coordinates are less than 1 and $U \subset \mathbb{R}^n - \{p_0, p_1\}$ is all points whose first coordinates are greater than 0.]

P.S. It might also be worthwhile to take this further and prove that $\mathbb{R}^n - \{p_0, \cdots, p_k\}$ is also simply connected for $k \geq 2$.