Mathematics 205A, Fall 2014, "Examination 4"

Answer Key

1. (*i*) If A is a retract of X and $p \in A$, then the induced map of fundamental groups $\pi_1(A,p) \to \pi_1(X,p)$ is 1–1. If $\mathbb{RP}^1$ is homeomorphic to a retract of $\mathbb{RP}^n$ if $n \geq 2$, this means that $\pi_1(\mathbb{RP}^1)$ is isomorphic to a subgroup of $\pi_1(\mathbb{RP}^n)$. Since the first group is infinite cyclic and the second is cyclic of order 2, we know that $\pi_1(\mathbb{RP}^1)$ is not isomorphic to a subgroup of $\pi_1(\mathbb{RP}^n)$, and this yields the conclusion in this part of the problem.■

(*ii*) First of all, there are two points in the inverse image of $f^{-1}(z_0)$ because there are two distinct square roots for every nonzero complex number, and if z_0 has unit length then the two roots w_j also have unit length (note that $1 = |z_0| = |w_j^2| = |w_j|^2$).

Regarding the main part of the problem, let $p : \mathbb{R} \to S^1$ be the usual covering space projection, and let $z_0 = p(t_0)$. If $0 < s - u < 1$ let $\mathbf{ARC}(s,u)$ denote the open arc which is the image of the open interval (s,u) under p. Then one can check directly that the inverse image of the open semicircle $\mathbf{ARC}\left(t_0 - \frac{1}{4}, t_0 + \frac{1}{4}\right)$ is the union of the two open quarter circles $\mathbf{ARC}\left(\frac{1}{2}t_0 - \frac{1}{8}, t_0 + \frac{1}{8}\right)$ and $\mathbf{ARC}\left(\frac{1}{2}t_0 + \frac{3}{8}, t_0 + \frac{5}{8}\right)$. — Geometrically, the inverse image of a 90° arc centered at $p(t_0)$ is a pair of 45° arcs which are centered at $p(\frac{1}{2}t_0)$ and $-p(\frac{1}{2}t_0) = p(\frac{1}{2}t_0 + \frac{1}{2})$.■

2. Follow the hint, and start with the free group F on k generators. Let N be the normal subgroup which is (normally) generated by all elements of the form x^n, where n runs through all the elements of F, and define G to be the quotient F/N. Then by construction G has k generators, and $g^n = 1$ for all $g \in G$.

Suppose now that H is an arbitrary group such that H has k generators and $h^n = 1$ for all $h \in H$. Then there is a homomorphism $\varphi_0 : F \to H$ which sends the free generators of F to a set of k generators for H. It follows that φ_0 is onto because its image contains a set of generators for H; let L be the kernel of φ_0. Since $h^n = 1$ for all $h \in H$ and φ_0 is onto, it follows that if $x \in F$ then $x^n \in L$. Now N is the unique minimal normal subgroup containing all the powers x^n, and therefore we have $N \subset L$, which implies that

$$H \;\cong\; F/L \;\cong\; (F/N)\;/\;(L/N)$$

is isomorphic to a quotient group of $G = F/N$.■

Note. Chapter 18 of Hall, *Theory of Groups* (MacMillan, 1959), provides further information on the Burnside Problem mentioned in the examination itself.

3. (*i*) Use the Seifert-van Kampen Theorem. The latter implies that $\pi_1(X,p)$ is isomorphic to the free product of G and $\{1\}$ modulo the normal subgroup (normally) generated by all elements of the $i(h) \cdot j(h)^{-1}$ where $i : H \subset G$ and $j : H \to \{1\}$ is the trivial homomorphism. Since j is trivial, the normal subgroup is actually generated by the elements $i(h)$, and under the standard identification of $G * \{1\}$ with G, this subgroup corresponds to H. Therefore $\pi_1(X,p)$ is isomorphic to G/H.■

(*ii*) Once again, use the Seifert-van Kampen Theorem, and also follow the hint. Recall that U and V are the open subsets of $\mathbb{R}^n - \{p_0, p_1\}$ defined by the first coordinate inequalities $x_1 < 1$ and $x_1 > 0$ respectively.

Since $U \cap V = (0,1) \times \mathbb{R}^{n-1}$ and the latter is contractible, the Seifert-van Kampen Theorem implies that $\pi_1(\mathbb{R}^n - \{p_0, p_1\}, p_{1/2})$ is isomorphic to the free product of $\pi_1(U, p_{1/2})$ and $\pi_1(V, p_{1/2})$.

Therefore it is enough to show that both of these groups are trivial. In both cases this can be done by the same sort of proof which shows that $\pi_1(\mathbb{R}^n - \{p\})$ is trivial for each $p \in \mathbb{R}^n$ (approximate a closed curve by a broken line curve, and show that the image of this curve lies in a finite union of hyperplanes).■

P.S. We can prove that that $\mathbb{R}^n - \{p_0, \cdots, p_k\}$ is also simply connected by an inductive argument which is similar to the preceding solution.

For each j such that $j \leq k$ let U_j be the set of all points in $\mathbb{R}^n - \{p_0, \cdots, p_k\}$ such that $x_1 < k+1$, and let V be the set of all points whose first coordinates satisfy $x_1 > k-1$. The solution to (ii) implies that U_0 and V_1 are simply connected. Similarly, if V_k is the set of all points in $\mathbb{R}^n - \{p_0, \cdots, p_k\}$ such that $x_1 > k-1$, then the same reasoning as in the solution to (ii) implies that V_k is simply connected.

Consider the following statements, which are meaningful for all $k \geq 1$:

(A_k) The open set U_j is simply connected for all $j < k$.

(B_k) The open set $\mathbb{R}^n - \{p_0, \cdots, p_k\}$ is simply connected.

The solution to (ii) shows that (A_1) and (B_1) are true, so we have the starting point for an inductive argument.

We shall first prove that (A_k) implies (B_k) for all k. This follows as in the solution to (ii), for we know that

$$\mathbb{R}^n - \{p_0, \cdots, p_k\} \quad = \quad U_k \cup V_k\,, \quad (k-1, k+1) \times \mathbb{R}^{n-1} \quad = \quad U_k \cap V_k$$

and hence $\mathbb{R}^n - \{p_0, \cdots, p_k\}$ is a union of two simply connected open subsets whose intersection is arcwise connected. Therefore $\mathbb{R}^n - \{p_0, \cdots, p_k\}$ is simply connected by the Seifert-van Kampen Theorem.

Next, we shall prove that (A_k) implies (A_{k+1}) for all k. Let W_k be the set of points in $\mathbb{R}^n - \{p_0, \cdots, p_k\}$ whose first coordinates satisfy $k-1 < x_1 < k+1$; once again the argument in the solution to (ii) implies that W_k is simply connected. Then we have

$$U_{k+1} \quad = \quad U_k \cup W_k`, \quad (k-1, k+1) \times \mathbb{R}^{n-1} \quad = \quad U_k \cap W_k$$

and by the same reasoning as in the preceding paragraph we see that U_{k+1} is also simply connected.■