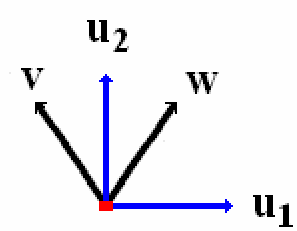
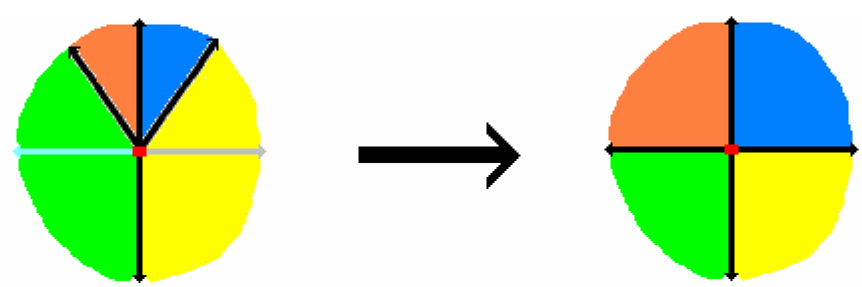
A Flattening Construction

In the discussion below, all vectors are assumed to lie in the coordinate plane.

We are given two unit vectors $\mathbf{v}$ and $\mathbf{w}$ that are linearly independent, and $\mathbf{u}_2$ is a unit vector which points in the same direction as $\mathbf{v} + \mathbf{w}$. The drawing below suggests that we can find a vector $\mathbf{u}_1$ in the span of $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{u}_1$ is perpendicular to $\mathbf{u}_2$ and the inner product of $\mathbf{u}_1$ and $\mathbf{w}$ is positive; a rigorous proof is given in the commentaries. Since $\mathbf{v} + \mathbf{w}$ is a multiple of $\mathbf{u}_2$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{u}_1$, it follows that the inner product of $\mathbf{v}$ and $\mathbf{u}_1$ must be negative (note that the inner product of $\mathbf{v} + \mathbf{w}$ and $\mathbf{u}_1$ is zero). Also, since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent it follows that their inner product lies in the open interval $(-1, 1)$, and this in turn implies that the inner product of $\mathbf{u}_2$ (a positive multiple of $\mathbf{v} + \mathbf{w}$) with both $\mathbf{v}$ and $\mathbf{w}$ must be nonzero because the inner products of $\mathbf{v}$ and $\mathbf{w}$ with $\mathbf{v} + \mathbf{w}$ are both equal to $1 + \langle \mathbf{v}, \mathbf{w} \rangle$.



Suppose now that $\mathbf{u}_1$ and $\mathbf{u}_2$ are the standard unit vectors in the coordinate plane. Then there is a homeomorphism of this plane to itself which maps the vertical axis to itself by the identity and flattens the angle $\mathbf{v}\,\mathbf{0}\,\mathbf{w}$ (where $\mathbf{0}$ is the origin) into the horizontal axis. This is illustrated by the drawing below, in which regions of the same color correspond under the homeomorphism.



As usual, it is necessary to express the homeomorphism formally. Actually, the inverse homeomorphism is easier to express in terms of equations, so we shall describe the latter instead. Given a point (x, y) in the coordinate plane, its image under the inverse is equal to $x\,\mathbf{w} + y\,\mathbf{u}_2$ if x is nonnegative and $|x|\,\mathbf{v} + y\,\mathbf{u}_2$ if x is nonpositive.

Commentaries on this construction

There are several things that need to be checked in the discussion on the first page.

First of all, suppose that we are given two linearly independent vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$ such that $|\mathbf{v}| = |\mathbf{w}| = 1$. The assertion in line 5 amounts to saying that there is an orthonormal basis $\{\mathbf{u}_1,\ \mathbf{u}_2\}$ for $\mathbb{R}^2$ such that $\mathbf{u}_2$ is a positive multiple of $\mathbf{v}-\mathbf{w}$ and $\langle \mathbf{u}_1, \mathbf{w}\rangle$ is positive. In fact, a nonzero vector perpendicular to $\mathbf{w}-\mathbf{v}$ is given by $\mathbf{v}+\mathbf{w}$ because $\mathbf{v}$ and $\mathbf{w}$ both have unit length and they are linearly independent (compute the dot product and observe that it is zero). Therefore we can obtain the desired orthonormal basis if we multiply the sum and difference vectors by appropriate scalars.

Secondly, in the final sentence of the preceding page we gave a formula for the inverse map to the homeomorphism we wanted to construct. In order to check the formula for the inverse is a well defined function, we need to verify that the two definitions agree when $x = 0$. However, in this case both formulas yield the value $y\mathbf{u}_2$, so there is no problem.

Finally, to be complete we should also check that the inverse mapping from on the preceding page actually has a continuous inverse. By the discussion on this page we have $\mathbf{w} = a\,\mathbf{u}_1 + b\,\mathbf{u}_2$, where $a > 0$ Similarly, we have $\mathbf{v} = c\,\mathbf{u}_1 + d\,\mathbf{u}_2$, and since $\mathbf{v}-\mathbf{w}$ is a positive multiple of $\mathbf{u}_1$ we know that $b = d$ and $a - c > 0$. But we also know that $\mathbf{v}+\mathbf{w}$ is a positive multiple of $\mathbf{u}_2$, which means that $a + c = 0$ and $b + d > 0$. If we combine these we see that $c = -a$ and $b + d = 2b > 0$; the latter then implies that $d = b > 0$. These formulas (and the linear independence of $\mathbf{u}_1$ and $\mathbf{u}_2$) imply that the inverse mapping $(s, t) = \psi{-}1(x, y)$, which maps the picture on the right to the picture on the left, should be given by

$$\begin{array}{rl} xa\,\mathbf{u}_1 + xb\,\mathbf{u}_2 + y\,\mathbf{u}_2 & \text{if} \quad x \geq 0, \\ -|x|a\,\mathbf{u}_1 + |x|b\,\mathbf{u}_2 + y\,\mathbf{u}_2 & \text{if} \quad x \geq 0. \end{array}$$

These expressions are suppposed to be equal to $s\,\mathbf{u}_1 + t\,\mathbf{u_2}$, so in order to show that we have a continuous inverse, we need only set the displayed expressions equal to this linear combination, solve for x and y in terms of s and t, and check that x and y are continuous functions of s and t. If $x \leq 0$ then $xa = -|x|\,a$, so the for all choices of x we have $s = xa$ or equivalently $s = x/a$. One particular consequence of this is that $s \geq 0$ if and only if $x \geq 0$. Now we also have $t = y + xb$ if $x \geq 0$ and $t = y + |x|b$ if $x \leq 0$, and this simplifies to $t = y + |x|b$ for all x. Combining these, we have $t = y + (|s|b/a)$ for all x, so that $y = t - (|s|b/a)$. These formulas imply that ψ is 1–1 and has a continuous inverse. But this inverse is just the flattening map, and therefore it follows that the flattening map defines a homeomorphism from $\mathbb{R}^2$ to itself which sends the angle $\angle\mathbf{w0v}$ to the $\mathbf{u}_1$-axis.