Functors and Isomorphisms

Thm. Let $F: \mathcal{A} \to \mathcal{B}$ be a $\begin{Bmatrix} \text{contravariant} \\ \text{covariant} \end{Bmatrix}$ functor, and let $g: X \to Y$ be an isomorphism in $\mathcal{A}$. Then $F(g)$ is an isomorphism in $\mathcal{B}$.

Derivation(s): Let $h: Y \to X$ be the inverse morphism s.t. $h \circ g = id_X$, $g \circ h = id_Y$.

COVARIANT CASE. We have

$$\begin{Bmatrix} id_{F(X)} = F(id_X) = F(h \circ g) = F(h) \circ F(g), \\ id_{F(Y)} = F(id_Y) = F(g \circ h) = F(g) \circ F(h) \end{Bmatrix}.$$

Therefore $F(g)$ is an isomorphism, and $F(h)$ is its inverse.

CONTRAVARIANT CASE. Almost the same, but now

$$\begin{Bmatrix} F(h \circ g) = F(g) \circ F(h), \\ F(g \circ h) = F(h) \circ F(g) \end{Bmatrix}.$$

In this case we conclude that $F(g): F(Y) \to F(X)$ is an isomorphism, and $F(h)$ is its inverse. ∎