Introduction

This is the second part of the first entry level graduate courses in topology and geometry, and the goal is to develop algebraic techniques for analyzing closed curves in a topological space. More detailed discussions of the motivation and approach are given below and at the beginning of Unit VII (the first unit in this part of the course). The numbering of these notes is a continuation of the numbering in the notes for the first part of the course (the file `gentopnotes2014.pdf`); for example, Theorem VI.7.8 would refer to Theorem 8 in Section VI.7 of the cited document (however, there is no such section in those notes).

The basic texts for this portion of the course are the following:

[**M**] **J. R. Munkres.** *Topology* (Second Edition), *Prentice-Hall, Saddle River NJ*, 2000. ISBN: 0–13–181629–2. [This is the text for the previous course in the sequence.]

[**H**] **A. Hatcher.** *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0–521–79540–0.

This book can be legally downloaded from the Internet at no cost for personal use, and here is the link to the online version:

`www.math.cornell.edu/~hatcher/AT/ATpage.html`

The material in this part of the course has become fairly standard, and it is directly related to some phenomena involving line integrals and functions of a complex variable. One major theme is the creation of algebraic "pictures" of a topological space which are obtained by studying certain types of topological configurations in the space. Ever since (at least) the beginning of the 17^{th} century, mathematicians and others have recognized the effectiveness and power of algebraic techniques for analyzing geometrical problems by transforming geometric input into algebraic terms, solving the associated algebraic questions, and translating the algebraic results back into the original geometric setting. The central concept in the second part of the course is the *fundamental group* of a space, which is an algebraic object constructed from the 1-dimensional configurations given by closed curves which start and end at a fixed basepoint.

One way to compare the two parts of the course is to describe the conclusions which follow from the respective methods: Using point set topology one can show that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic if $n \geq 2$, and using point fundamental groups, one can show that $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic if $n \geq 3$. — In the second course of the geometry/topology sequence, still other methods are developed to prove that $\mathbb{R}^m$ and $\mathbb{R}^n$ are never homeomorphic if $m \neq n$. Although this may seem obvious intuitively, it is still necessary to give a formal proof because intuition can be misleading.

Taking all things into account, the first part of Munkres (on point set topology) is one of the very best accounts of the subject, with an excellent balance of clear exposition, logical completeness and drawings to motivate the underlying geometrical content of the subject (there are some peculiar choices of terms and symbolism, and in a number of instances more motivation would help, but the perfect text is an ideal which is rarely if ever realized). The second part of Munkres, which includes the material on fundamental groups, comes close to meeting this standard. However, there are numerous cases where more motivational comments and drawings would help, and sometimes the logical thoroughness of the exposition interferes with its clarity. To its credit, the second part gives logically complete accounts of several basic applications of topology to basic geometrical results like the Fundamental Theorem of Algebra and the Jordan Curve Theorem (a simple closed curve in the plane separates it into two connected pieces), but the proofs really push the theory in the book to its limits, and consequently the reasoning is often very delicate and difficult to follow. We shall see that homology theory often yields much simpler and more conceptual proofs.

Hatcher's book begins by covering the same topics which appear in the second half of Munkres, and it proceeds to go much further in the subject. The challenges faced in covering the further material are much greater than the corresponding challenges in Munkres. In particular, the gap between abstract formalism and geometrical intuition is much greater, and it is not clear how well any single book can reconcile these complementary factors. More often than not, algebraic topology books stress the former at the expense of the latter, and one important strength of Hatcher's book is that its emphasis tilts very much in the opposite direction. The book makes a sustained effort to include examples that will provide insight and motivation, using pictures as well as words, and it also attempts to explain how working mathematicians view the subject. Because of these objectives, the exposition in Hatcher is significantly more casual than in many books on the subject. Unfortunately, the book's informality is arguably taken too far in numerous places, leading to significant problems in several directions; these include assumptions about prerequisites, clarity, wordiness, thoroughness and some sketchy motivations that are difficult for many readers to grasp.

Prerequisites

This part of the course is based upon material developed in the first part. There is also some background material which is needed here but does not appear in the first part of the course.

Algebra

Some concepts in group theory are needed; most are at the undergraduate level. Several other concepts from group theory are presented in Munkres and will be covered in the course. Material from standard undergraduate linear algebra courses will also be used as needed. Everything we need can be found in the following standard graduate algebra textbook:

> **T. Hungerford.** *Algebra.* (Reprint of the 1974 original edition, Graduate Texts in Mathematics, No. 73.) *Springer-Verlag, New York–Berlin–etc.*, 1980. ISBN: 0–387–90518–9.

At some points in this course we shall invoke the following basic result, which is proved in graduate level algebra courses (for example, Sections II.1 and II.2 of Hungerford).

STRUCTURE THEOREM FOR FINITELY GENERATED ABELIAN GROUPS. *Let G be a finitely generated abelian group (so every element can be written as a monomial in integral powers of some finite subset $S \subset G$). Then G is isomorphic to a direct sum*

$$(H_1 \oplus \cdots \oplus H_b) \oplus (K_1 \oplus \cdots \oplus K_s)$$

where each H_i is infinite cyclic and each K_j is finite of order t_j such that t_{j+1} divides t_j for all j. — For the sake of uniformity set $t_j = 1$ if $j > s$. Then two direct sums as above which are given by $(b; t_1, \cdots)$ and $(b'; t'_1, \cdots)$ are isomorphic if and only if $b = b'$ and $t_j = t'_j$ for all j.

For the purposes of this course, it is enough to understand the statement of the structure theorem; the proof itself is not part of the course or its prerequisites.

Analysis

We shall assume the basic material from an upper division undergraduate course in real variables as well as material from a lower division undergraduate course in multivariable calculus through the theorems of Green and Stokes as well as the 3-dimensional Divergence Theorem. The classic text by W. Rudin (*Principles of Mathematical Analysis*, Third Edition) is an excellent reference for real variables, and the following multivariable calculus text contains more information on the that subject than one can usually find in the usual 1500 page calculus texts (unfortunately, this book is far from perfect, but especially at the graduate level it may be useful for review purposes). Clearly there are also many other sources for this material; the main point is that a reader might have to refresh his or her memory on a few topics at some points in the notes.

> **J. E. Marsden and A. J. Tromba.** *Vector Calculus* (Fifth Edition), *W. H. Freeman & Co., New York NY*, 2003. ISBN: 0–7147–4992–0.

The concept of a **category** does not appear explicitly in Munkres, but it is implicit in many places, and at numerous points in these notes it will be useful to formulate things using categories as part of the framework. We have attempted to limit this usage to situations where the formalism seems to simplify the discussion, so only need a few basic ideas are needed, and we shall summarize here the concepts that appear almost immediately in the notes. The course directory file

`categories2014.pdf`

(which is 10 pages long) gives a more organized treatment of the topics discussed here, and **reading the document is an implicit assignment for the course.**

A **category** is an abstract mathematical system which reflects some very basic features of many classes of mathematical objects and the well-behaved morphisms relating them. In set theory the objects and morphisms are sets and functions of sets, and in topology the most basic examples involve topological spaces and continuous mappings. There are many algebraic examples, including groups and morphisms, vector spaces over a fixed field $\mathbb{F}$ and $\mathbb{F}$-linear transformations, or partially ordered sets and monotonically increasing functions. Many other examples appear in the file cited above.

In addition to a family of **objects** as above, the data for a category also include **morphisms** from one object to another with specified objects as their *domains* (sources) and *codomains* (targets); a morphism together with its domain and codomain are generally denoted by notation like $f : A \to B$. There are also binary algebraic operations defined for certain pairs of morphisms, and these behave formally like composition of functions in the following respects:

(i) The composition $g \circ f$ of g and f is defined if and only if the target of f is the source of g.

(ii) For each object X there is an "identity morphism" $1_X : X \to X$ (sometimes we call this map id_X), and for each morphism $f : A \to B$ we have $1_B \circ f = f = f \circ 1_A$.

(iii) There is an associative law $h \circ (g \circ f) = (h \circ g) \circ f$ for threefold compositions.

The most important additional concept is that of an **isomorphism** between two objects A and B. This involves a pair of morphisms $f : A \to B$ and $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. As elsewhere in mathematics, if one has such a pair of morphisms we say that f and g are inverse to each other (or inverses of each other).

The ubiquity of categories reflects a basic fact: *If a class of mathematical objects is defined, it is usually possible to define a good concept of mappings or morphisms from one object to another without too much trouble.* — From this perspective, it is natural to speculate about an appropriate notion of morphism relating one category to another. It turns out that there are **two** such notions called **contravariant functors** and **covariant functors**. A *covariant functor* is a system of transformations such that

(a) for each object X in the source category there is an associated object $T(X)$ in the target category,

(b) for each morphism $f : X \to Y$ in the source category there is an associated morphism $T(f) : T(X) \to T(Y)$ in the target category,

(c) the construction on morphisms preserves identity morphisms and compositions; the latter means that $T(g \circ f) = T(g) \circ T(f)$.

Here is an example involving topological spaces: If X is a topological space, let $T(X)$ be the set of continuous curves $\gamma : [0,1] \to X$, and if $f : X \to Y$ is continuous define $T(f)\gamma = f \circ \gamma$. The fundamental group of a pointed space is a more sophisticated example of this sort going from pointed topological spaces to groups.

As noted above, there is also a dual concept of *contravariant functor* from one category to another; the main differences with covariant functors are that a morphism $f : A \to B$ is sent to $T(f) : T(B) \to T(A)$ (*i.e.*, the domain and codomain are switched) and the composition identity is $T(g \circ f) = T(f) \circ T(g)$ (*i.e.*, the order of composition is reversed).

One basic example of a contravariant functor is the dual space construction on a category of vector spaces over some field $\mathbb{F}$. Specifically, a vector space V is sent to the space V^* of $\mathbb{F}$-linear functionals $V \to \mathbb{F}$, and if $T : V \to W$ is a linear transformation then $T^* : W^* \to V^*$ sends a linear functional $f : V \to \mathbb{F}$ to the composite $T \circ f$.

Functors have a simple but far-reaching property which is fairly easy to prove: *If the morphisms f and g are inverse to each other and T is a functor (covariant or contravariant), then $F(f)$ and $F(g)$ are also inverse to each other.*

Since functors are mathematical objects, one can speculate even further about morphisms relating functors and whether such a notion is more than a formal curiosity. It turns out that there is an extremely useful notion called a **natural transformation** of functors (where both the source and target have the same variance). Since this concept is not needed until later in the course sequence, we shall pass on discussing it here.