Introduction

These first entry level graduate course in topology and geometry splits naturally into two parts:

1. Topics from point set (also known as *general*) topology.

2. Fundamental groups and covering spaces; only part of this subject is covered in the first course, and the remainder is covered in the second course.

These notes cover material for the first part of the course. The corresponding notes for the second part of the course are in the document `fundgp-notes.pdf`.

The main text for both parts of the course is the following classic book on the subject:

J. R. Munkres. *Topology* (Second Edition), *Prentice-Hall, Saddle River NJ*, 2000. ISBN: 0–13–181629–2.

The choice of topics to be covered

Most beginning graduate students have seen at least some material from point set topology in an undergraduate real variables course, and some have seen even more material in undergraduate topology courses, but the topics vary depending upon the institution and instructor. Therefore these notes develop the subject from the beginning for the sake of completeness, but the course itself will spend little if any time on some topics widely covered in undergraduate real variables courses, and coverage of a few other topics will focus on the generalizations of other key concepts and results to abstract topological spaces. Sections of the notes with limited or no course coverage are flagged in the notes with numerical superscripts which have the following meanings:

Section name$^{(3)}$ Some material from undergraduate real variables courses that will be skipped in the course itself. However, everything in this section is part of the material covered in course and qualifying examinations.

Section name$^{(2)}$ Material probably seen in previous courses, only covered lightly to provide a basis for material in the course itself; everything in this section is also part of the material covered in course and qualifying examinations.

Section name$^{(1)}$ Prerequisite material from set theory or real variables courses, not covered in the course or qualifying examinations. Section III.3 is an exception; the material up to, but not including, the subheading *Baire spaces* is part of the material covered in course and qualifying examinations.

Section name$^{(0)}$ Not covered in the course or qualifying examinations, included mainly for reference purposes. Section VI.5 is an exception to this rule; the statements of the main theorems in this section should be understood, but there is no need to know the proofs.

Numbering conventions for results

Notation like "Theorem X.11.22" will indicate Theorem 22 in Section X.11 of the notes (but the specific citation in this sentence does not correspond to anything in the notes — there is no Unit X).

Some alternate and additional references

The online directory

`http://math.ucr.edu/~res/math145A-2014/`

contains extensive material for a point set topology course at the undergraduate level and may be useful as a source of information on basic topics which are either covered lightly or not at all in the present course. In particular, the file

`http://math.ucr.edu/~res/math145A-2014/geom-background.pdf`

includes links to undergraduate mathematics courses whose contents led to the development of material in this course.

The books by Dugundji and Kelley in the bibliography are excellent graduate level point set topology texts, and each of these views the subject somewhat differently from the perspective in Munkres and these notes. The following more recent text is also a very good alternate reference for much of the material in this course:

> **T. Lawson.** *Topology: A Geometric Approach. Oxford University Press, New York–etc.*, 2003. ISBN: 0-19-920248-6.

Finally, we should also mention the undergraduate level textbook around which the files in the directory `http://math.ucr.edu/~res/math145A-2014/` are organized.

> **W. A. Sutherland.** *Introduction to Metric and Topological Spaces.* (Second Ed.) *Oxford University Press, New York–etc.*, 2009. ISBN: 019956308X.

There is also a very useful companion website for this book:

`http://www.oup.com/uk/booksites/content/9780199563081/`

Among other things, this site includes topics and details not in Sutherland's book, and there are also solutions to half of the exercises.

NOTE ON WEB LINKS. Many of the links to the Word Wide Web in these notes are so long and complicated that they can be frustrating to enter from a keyboard, but unfortunately some links are not clickable due to software constraints; if such cases, there is a clickable link in the file `math205Awww2014.pdf`.

References to Sutherland for these notes

For several reasons such as review or slightly different approaches, some students might find it helpful to use Sutherland's book as a supplementary reference for this course. Therefore we are including a table which gives the chapters in Sutherland (and the notes on these chapters in the files `http://math.ucr.edu/~res/math145A-2014/math145Anotes*.pdf` where the numbers in the wild card string $*$ refer to one or more chapters in Sutherland) which correspond to the appropriate sections of these notes; sections with little or no corresponding material in Sutherland are not listed in the table.

Section in these notes	*Chapter(s) in Sutherland*
Introduction	1 (and Prefaces)
I.1	2
I.2	2 – 3
I.4	4
II.1	5 – 8, 11, 16
II.2	6 – 9
II.3	5 – 8
II.4	5, 10
III.1	13
III.2	16 – 17
III.3	17
III.4	12
III.5	12
V.1	15
VI.1	8
VI.2	13 – 14
VI.3	7, 11 – 12
VI.5	11

Secondary course goals

We have already noted that all students in an entry level graduate course have seen simplified versions of many (if not most) concepts and results from the first part of Mathematics 205A in prerequisite courses. Some of the reasons for the organization of the course go beyond the introduction of new material and the coverage of many topics in greater detail than in the prerequisites. These include (1) the presentation of an abstract, unified approach to some basic topics in mathematics, (2) illustrations that increasing formality and abstraction sometimes yield simpler and more conceptual understandings of basic facts, (3) improvement of skills in reading and writing proofs.

The enhancement of proof writing skills is particularly important, and it is indispensable for studying mathematics at the graduate level. Many proofs at this level are significantly more difficult than their counterparts at the undergraduate level. In particular, the level of abstraction is usually higher, the arguments are frequently much longer or more complicated, simple steps in arguments are sometimes omitted or mentioned only briefly, and often the approaches are far less direct, in many cases with crucial steps relying on points which at first are easily overlooked.

A review of mathematical proofs at the undergraduate level is given in the course directory file `mathproofs.pdf`, and a few additional suggestions are given in the file `math205Asolutions00.pdf`. Of course, many other articles on writing mathematical proofs can be found by searching for phrases like *writing proofs* and/or *writing proofs topology* using **Google** or a similar search engine. General comments about such searches appear in the file `aabInternetresources.pdf`.

Remarks on priorities

In order to write mathematical proofs at the graduate level, *it is necessary to understand most proofs in the texts and course notes so thoroughly that they can be explained convincingly to someone else who has adequate background knowledge.* However, there are proofs for which a more passive understanding of the proof is enough; in other words, the reader should be understand the assertions well enough to conclude whether or not they are valid, but for a various reasons there is no need to have an active understanding as described in the previous sentence.

Such proofs already arise in prerequisite courses, and one example involves the existence and uniqueness (up to order-preserving algebraic isomorphism) of a system satisfying the standard axioms for the real number system. Without these existence and uniqueness results, most of mathematics would not have an acceptable logical foundation. However, in each case there are reasons why it is not that important to dwell too much upon the proofs. There are two standard proofs of existence, one due to R. Dedekind and the other due to G. Cantor. In each case, there is a formal construction based upon a reasonable idea (for the Dedekind approach it is the idea that a real number is specified by the rational numbers which are strictly less than it, and for the Cantor approach it is the idea that a real number is the limit of a sequence of rational numbers), but the verifications are generally tedious and the methods of proof are not useful for much else. One might compare the details of verification to the scaffolding in a large construction project: The structure is absolutely necessary, but it can be set aside once everything is completed. The following quote from the Sherlock Holmes story, *A Study in Scarlet* by A. C. Doyle (1859–1930), reflects the need to leave some details in the background.

> ··· My [Watson's] surprise reached a climax, however, when I found incidentally that he [Holmes] was ignorant of the Copernican Theory and of the composition of the Solar System. That any civilized human being in this nineteenth century should not be aware that the earth traveled round the sun appeared to be to me such an extraordinary fact that I could hardly realize it.
>
> "You appear to be astonished," he said, smiling at my expression of surprise. "Now that I do know it I shall do my best to forget it."
>
> "To forget it!"
>
> "You see," he explained, "I consider that a man's brain originally is like a little empty attic, and you have to stock it with such furniture as you choose. A fool takes in all the lumber of every sort that he comes across, so that the knowledge which might be useful to him gets crowded out, or at best is jumbled up with a lot of other things so that he has a difficulty in laying his hands upon it. Now the skilful workman is very careful indeed as to what he takes into his brain-attic. He will have nothing but the tools which may help him in doing his work, but of these he has a large assortment, and all in the most perfect order. It is a mistake to think that that little room has elastic walls and can distend to any extent. Depend upon it there comes a time when for every addition of knowledge you forget something that you knew before. It is of the highest importance, therefore, not to have useless facts elbowing out the useful ones."
>
> "But the Solar System!" I protested"
>
> "What the deuce is it to me?" he interrupted impatiently; "you say that we go round the sun. If we went round the moon it would not make a pennyworth of difference to me or to my work."

We should add an important qualification to this analogy. For the sorts of mathematical proofs we describe, it is highly desirable to know enough about the proof that one can summarize the main steps in the argument to someone with adequate background. In the case of the existence and uniqueness proofs for the real numbers, this may be done as follows:

> **Existence.**
> The Cantor or Dedekind construction of objects which correspond to our intuitive idea of how real numbers can be specified.
> Define reasonable candidates for addition, multiplication and ordering of the constructed objects.
> Show that these defined notions satisfy the axioms for a real number system.
>
> **Uniqueness.**
> Define the map from one system to another starting by sending the zero and unit to one into the zero and unit of the other.
> Recursively define a map φ from the "positive integers" in one system to the "positive integers" in the other by sending $n + 1$ times the unit in the first system to $\varphi(n)$ + unit in the second.
> Prove that φ is 1–1 onto, order preserving, and also preserves sums and products.
> Extend φ to a map from the "rational numbers" in one system to the "rational numbers" in the other, and prove that this extension is also 1–1 onto, order preserving, and also preserves sums and products.
> Extend φ to a map from the entire first system to the entire system using the fact that an element of a complete ordered field is determined by the "rational numbers" which are strictly less than it.
> Verify that this extension is 1–1 onto, order preserving, and also preserves sums and products.

The preceding should also apply to several basic results (*e.g.*, Tychonoff's Theorem in Section III.1 and several results in Section VI.5) which are stated without proof in the notes.