Note on the Hilbert Cube

The *Hilbert cube* **HQ** is defined to be a cartesian product of $\aleph_0$ copies of the unit interval $[0, 1]$. This space was also mentioned in Appendix C to `gentopnotes2014.pdf`. By the Tychonoff Theorem this space is compact, and it is also metrizable because a countably infinite product of metrizable spaces is always metrizable. In Appendix C we noted that **HQ** is *homogeneous* — in other words, given two points in this space there is a homeomorphism sending the first point to the second — and this contrasts sharply with the fact that all finite products $[0, 1]^n$ are not homogeneous (for example, a homeomorphism of $[0, 1]^n$ to itself cannot send the point whose coordinates are all zero to the point whose coordinates are all equal to $\frac{1}{2}$; see the previously cited Appendix C for more on this). The Hilbert Cube has been studied extensively, both for its own sake and for its role in solving several problems of independent interest (for example, see the article by T. Chapman in the bibliography of `gentopnotes2014.pdf`), but we shall not try to elaborate on any of this here. Our objective in this document is merely to prove the following property of the Hilbert Cube.

PROPOSITION. *There is no nonnegative integer n such that the Hilbert Cube* **HQ** *is a topological n-manifold.*

Intuitively this may seem obvious, for one thinks informally of topological n-manifolds as finite dimensional objects, and **HQ** seems like it should be an infinite dimensional object. One obvious weakness of this intuition is that we have not defined a suitable notion of dimension for topological spaces. However, it turns out that one can define a reasonably well-behaved dimension, at least for topological spaces that are themselves reasonably well-behaved. There are a couple of approaches to doing this, and for most purposes the definition of choice in modern mathematics is the one which appears in Section 50 of Munkres (see page 305) and is generally known as the *Lebesgue covering dimension* to distinguish it from other approaches to defining topological dimensions for spaces. In fact, everything we need to know about topological dimension theory for the proof of the proposition can be found either in the text of Section 50 or in its exercises.

As in the previously cited Appendix C, we shall need to use a theorem from 205B called **Invariance of Domain**: *If U is open in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is continuous and $1-1$, then f is an open mapping.* — This result is Theorem VII.3.5 on pages 94–95 of the following notes for 205B:

`http://math.ucr.edu/~res/math205B-2012/algtop-notes.pdf`

And here is the specific input from Section 50 of Munkres that we shall need.

CLAIM. If X is a compact topological n-manifold, then the dimension of X (in the sense of Munkres) is less than or equal to n.

Proof of claim. By compactness and the definition of a topological manifold we know that X is a union of finitely many closed subsets B_k such that each is homeomorphic to the closed disk D^n. Theorem 50.6 of Munkres (see pp. 314–315) implies that $\dim B_k \leq n$ for all k. We can now apply Corollary 50.3 (see Munkres, p. 308) to conclude that $\dim X \leq n$.∎

Proof of the proposition. Some notation will be convenient: For each nonnegative integer n let $\mathbf{HQ}_n$ be the subspace of all points in **HQ** whose i^{th} coordinate vanishes for each $i \geq n$ (hence $\mathbf{HQ}_n$ is homeomorphic to $[0, 1]^n$).

Suppose that **HQ** is a topological n-manifold for some $n \geq 0$. By Corollary 50.8 on page 314 of Munkres, it then follows that **HQ** must be homeomorphic to a subset of $\mathbb{R}^{2n+1}$. Let

$f : \mathbf{HQ} \to \mathbb{R}^{2n+1}$ be a homeomorphism onto a compact subset. Also, let $h : [0,1]^{2n+2} \to \mathbf{HQ}_{2n+2}$ be the homeomorphism mentioned earlier, let W denote the image of $(0,1)^{2n+2}$ under h, and let W_0 denote the image of $(0,1)^{2n+1} \times \{\frac{1}{2}\}$. Then Invariance of Domain implies that $f(W_0)$ is open in $\mathbb{R}^{2n+1}$. On the other hand, W_0 is also nowhere dense in $\mathbf{HQ}$, and hence $f[W_0])$ must also be nowhere dense in $\mathbb{R}^{2n+1}$. In particular, the latter implies that $f(W_0)$ cannot be an open subset of $\mathbb{R}^{2n+1}$, yielding a contradiction. The latter means that the assumption that $\mathbf{HQ}$ is a topological n-manifold must be false. It follows that $\mathbf{HQ}$ is not a topological n-manifold for any choice of n.■

FINAL REMARKS. In fact, a second countable topological n-manifold is n-dimensional in the sense of Munkres (the compact case of this result is noted without proof in Munkres). This dimension identity follows from the results in Section III.5 of the following notes for an algebraic topology class which is a sequel to 205B:

`http://math.ucr.edu/∼res/math246A-2012/advancednotes2012.pdf`

With this dimension identity at our disposal, we can prove the infinite dimensionality of $\mathbf{HQ}$ as follows: Suppose that $\dim \mathbf{HQ} = m$ for some positive integer m. By Theorem 50.1 of Munkres (see p. 306) we know that $\dim \mathbf{HQ} \geq \dim \mathbf{HQ}_n$ for all n. If we take $m = n + 1$ we obtain a contradiction, and this means that the Hilbert Cube must be an infinite dimensional topological space.■

Further references for dimension theory

W. Hurewicz and H. Wallman. *Dimension Theory* (Revised Edition, Princeton Mathematical Series, Vol. 4). *Princeton University Press, Princeton*, 1996.

K. Nagami. *Dimension Theory* (with an appendix by Y. Kodama, Pure and Applied Mathematics Series, Vol. 37). *Academic Press, New York*, 1970.

J. Nagata. *Modern Dimension Theory* (Second Edition, revised and extended; Sigma Series in Pure Mathematics, Vol. 2). *Heldermann-Verlag, Berlin*, 1983.

`http://en.wikipedia.org/wiki/Lebesgue_covering_dimension`

`http://en.wikipedia.org/wiki/Dimension`

`http://en.wikipedia.org/wiki/Inductive_dimension`

FRACTAL DIMENSIONS. There are several notions of *fractal dimension* for subsets of $\mathbb{R}^n$ which depend on the way in which an object is embedded in $\mathbb{R}^n$ and not just the subset's underlying topological structure; for example, various standard types of nonrectifiable curves in the plane have fractal dimensions which are numbers strictly between 1 and 2. Such objects are interesting for a variety of reasons , but they are mostly beyond the scope of this course (however, see the comments below), so we shall simply give two online references here along with links to material from an undergraduate topology course:

`http://en.wikipedia.org/wiki/Fractal_dimension`

`http://www.warwick.ac.uk/∼masdbl/dimension-total.pdf`

Some topological concepts which are central to the study of fractals are considered in Exercise 6 on pages 1–2 of the online document

`http://math.ucr.edu/∼res/math145A-2014/exercises02w14.pdf`

and the solution to this exercise is written out on page 6 of the following companion document:

`http://math.ucr.edu/∼res/math145A-2014/solutions02w14.pdf`