Lipschitz functions — II

This file provides detailed information promised in the course directory file `lipschitz1.pdf`. At some points we shall need input from multivariable calculus.

Lipschitz conditions

The restriction of a smooth function (say of class $\mathcal{C}^r$) to a compact set satisfies a strong form of uniform continuity that generalizes the matrix inequality

$$|Ax| \quad \leq \quad \| A \| \cdot |x|$$

where $x \in \mathbb{R}^n$ us viewed as an $n \times 1$ column vector, A is an $m \times n$ matrix with real entries, and $\| A \|$ is a nonnegative constant called the *(matrix) norm* of A. For the sake of completeness, here is the formal definition. It is a fairly straightforward exercise to show that if A is a real $m \times n$ matrix then there is some positive constant $M > 0$ such that $|Ax| \leq M|x|$ for all column vectors x (for example, if L_j is the length of the $j^{\rm th}$ column vector then we can take M to be n times the maximum of the lengths L_j), and the matrix norm $\| A \|$ is merely the greatest lower bound of all such constants M.

THEOREM. *Let U be open in $\mathbb{R}^n$, let $f : U \to \mathbb{R}^m$ be a $\mathcal{C}^1$ function, and let $K \subset U$ be compact. Then there is a constant $B > 0$ such that*

$$|f(u) - f(v)| \;\leq\; B\,|u - v|$$

for all $u, v \in K$ such that $u \neq v$.

The displayed inequality is called a *Lipschitz condition* for f. This strong form of uniform continuity associates to each $\varepsilon > 0$ a corresponding δ equal to ε/B. An example of a function not satisfying any Lipschitz condition is given by $h(x) = \sqrt{x}$ on the closed unit interval $[0, 1]$ (use the Mean Value Theorem and $\lim_{t\to 0^+} h'(t) = +\infty$). Incidentally, the inverse of this map is a homeomorphism that does satisfy a Lipschitz condition (*e.g.*, we can take $B = 2$).

The inequality

$$\Big| u - v \Big| \quad \geq \quad \Big| |u| - |v| \Big|$$

for $u, v \in \mathbb{R}$ shows that $f(x) = |x|$ is a function that satisfies a Lipschitz condition but is not $\mathcal{C}^1$.

If f is a mapping from one metric space $(X, \mathbf{d}_X)$ to another metric space $(Y, \mathbf{d}_Y)$, then a *Lipschitz constant* for f on a subset $K \subset X$ (not necessarily compact) is a number $B > 0$ such that

$$\mathbf{d}_Y\,(\,f(u), f(v)\,) \quad \leq \quad B \cdot \mathbf{d}_X(u, v)$$

for all $u, v \in K$ such that $u \neq v$; if no subset is specified then then the understanding is that we are considering a Lipschitz condition on all of X. Note that even if Lipschitz constants exist, they are definitely not unique; if B is a Lipschitz constant for f on a set K and $C > B$, then C is also a Lipschitz constant for f on a set K.

By the remarks preceding the statement of the theorem, if f is a linear transformation defined by an $m \times n$ matrix A, then by definition the matrix norm of A is an optimal Lipschitz constant for f.

Proof of theorem. For each $x \in K$ there is a $\delta(x) > 0$ such that $N_{2\delta(x)}(x) \subset U$. By compactness there are finitely many points $x_1, \cdots, x_q$ such that the sets $N_{\delta(x_i)}(x_i)$ cover K. Let B_i be the maximum of $\| Df \|$ for $|y - x_i| \leq \delta(x_i)$. If $B_i = 0$ for all i then $Df = 0$ on an open set containing K and therefore f is constant on K, so that the conclusion of the theorem is trivial. Therefore we shall assume some $B_i > 0$ for the rest of the proof.

By the Mean Value Estimate we know that $y, z \in N_{\delta(x_i)}(x_i)$ implies that $|f(y) - f(z)| \leq B_i |y - z|$.

Let $\eta > 0$ be a Lebesgue number for the open covering of K determined by the sets $N_{\delta(x_i)}(x_i)$, and let $M \subset K \times K$ be the set of all points $(u, v) \in K \times K$ such that $|u - v| \geq \eta/2$. The function $\Delta(u, v) = |u - v|$ is continuous on $K \times K$, and consequently it follows that M is a closed and thus compact subset of $K \times K$. Consider the continuous real-valued function on M defined by

$$h(u, v) = \frac{|f(u) - f(v)|}{|u - v|} .$$

Since the denominator is positive on M, this is a continuous function and therefore attains a maximum value A.

Let B be the maximum of the numbers $A, B_1, \cdots, B_k$, and suppose that $(u, v) \in K \times K$. If $(u, v) \in M$, then by the preceding paragraph we have $|f(u) - f(v)| \leq A \cdot |u - v|$. On the other hand, if $(u, v) \notin M$, then $|u - v| < \eta/2$ and thus there some i such that $u, v \in N_{\delta(x_i)}(x_i)$. By the Mean Value Estimate we know that $|f(u) - f(v)| \leq B_i \cdot |u - v|$ in this case. Therefore $|f(u) - f(v)| \leq B \cdot |u - v|$ for all u and v.■

Two solved exercises on Lipschitz functions

1. (i) Suppose that X and Y are subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and that $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ are maps that satisfy Lipschitz conditions on X and Y respectively. Prove that the composite $g \circ f$ also satisfies a Lipschitz condition.

SOLUTION.

By our assumptions there are positive constants A and B such that $|f(x) - f(x')| \leq A \cdot |x - x'|$ and $|g(y) - g(y')| \leq B \cdot |y - y'|$. If we set $y = f(x)$ and $y' = f(x')$ in these formulas we obtain

$$|g \circ f(x) - g \circ f(x')| \quad \leq \quad B \cdot |f(x) - f(x')| \quad \leq \quad AB \cdot |x - x'|$$

which shows that $g \circ f$ satisfies a Lipschitz condition.■

(ii) Suppose that $X \subset \mathbb{R}^n$, and let $f, g : X \to \mathbb{R}^m$ and $h : X \to \mathbb{R}$ satisfy Lipschitz conditions. Prove that $f + g$ satisfies a Lipschitz condition and if X is compact then $h \cdot f$ also satisfies a Lipschitz condition. If $h > 0$ and X is compact, does $1/h$ satisfy a Lipschitz condition? Prove this or give a counterexample.

SOLUTION.

Let A and B be Lipschitz constants for f and g respectively. Furthermore, let C be a Lipschitz constant for h.

We verify first that $f + g$ satisfies a Lipschitz condition:

$$|[f + g](x) - [f + g](x')| = |f(x) + g(x) - f(x') - g(x')| \;\leq\; |f(x) - f(x')| + |g(x) - g(x')| \;\leq$$

$$A \cdot |x - x'| + B \cdot |x - x'| \quad = \quad (A + B)\,|x - x'|$$

In order to prove that $h \cdot f$ satisfies a Lipschitz condition we need to use the upper bounds for $|f|$ and $|h|$ which are guaranteed by compactness. Call these bounds P and Q respectively. We then have that

$$|h(x)f(x) - h(x')f(x')| \quad = \quad |h(x)f(x) - h(x)f(x') + h(x)f(x') - h(x')f(x')| \;\leq$$

$$|h(x)f(x) - h(x)f(x')| + |h(x)f(x') - h(x')f(x')| \;\leq\; Q \cdot |f(x) - f(x')| + P \cdot |h(x) - h(x')| \;\leq$$

$$QA\,|x - x'| + PC\,|x - x'| \quad = \quad (QA + PC)\,|x - x'|$$

and hence the product satisfies a Lipschitz condition.

Finally, $1/h$ does satisfy a Lipschitz condition, and here is the proof: In addition to the preceding let $\mathbf{m}$ be the minimum value of $|h|$ on X. Then we have

$$\left|\frac{1}{h(x)} \; - \; \frac{1}{h(x')}\right| =$$

$$\frac{|h(x') \; - \; h(x)|}{|h(x) \cdot h(x')|} \; .$$

The denominator is at least $\mathbf{m}^2$, and the numerator is at most $C \cdot |x - x'|$, and therefore $C\mathbf{m}^{-2}$ is a Lipschitz constant for $1/h$.■

(iii) Suppose that $X \subset \mathbb{R}^n$, and let $f : X \to \mathbb{R}^m$ be given. Prove that f satisfies a Lipschitz condition if and only if all of its coordinate functions do.

SOLUTION.

$(\Longrightarrow)$ If f satisfies a Lipschitz condition with Lipschitz constant A and f_i is the i^{th} coordinate function, then

$$|f_i(x) - f_i(y)| \;\leq\; |f(x) - f(y)| \;\leq\; A \cdot |x - y|$$

for all x and y.■

$(\Longleftarrow)$ If each coordinate function f_i satisfies a Lipschitz condition, then one can write $f = \sum_i f_i\,\mathbf{e}_i$ where $\mathbf{e}_i$ is the standard i^{th} unit vector; therefore the conclusion follows from the first part of the exercise and finite induction.■

2. In the notation of the preceding exercise, suppose that $X = A \cup B$ and that f is continuous and satisfies Lipschitz conditions on A and B as well as on an open neighborhood of $A \cap B$. Does f satisfy a Lipschitz condition on $A \cup B$? Prove this or give a counterexample. What happens if we assume A and B are compact? Justify your answer.

SOLUTION.

The answer to the first question is NO. Consider the function on $\mathbb{R} - \{0\}$ that is 1 for positive numbers and -1 for negative numbers. This satisfies a Lipschitz condition on A and B as well as an open neighborhood of $A \cap B = \emptyset$. However, if we take x and x' to be $\pm 1/n$ then $|f(x) - f(x')| = 2$ while $|x - x'| = 2/n$, and hence any Lipschitz constant for f on $A \cup B$ would have to be at least n for every positive integer n. Therefore f does not satisfy a Lipschitz condition on $A \cup B$.

The answer to the second question is yes. Let U be an open neighborhood of $A \cap B$ on which f satisfies a Lipschitz condition, and let K_0 be the associated Lipschitz constant. Similarly, let K_1 and K_2 be Lipschitz constants for f on A and B respectively.

Formally, there are initially 64 cases to consider depending upon whether or not u lies in A, B or U (a total of 8 possibilities), and likewise for v. However, many of the formal possibilities are inconsistent with the conditions that u and v belong to $A \cup B$. In particular, we cannot have $u \notin A$ and $u \notin B$ and we cannot have $u \in A$, $u \in B$, but $u \notin U \supset A \cap B$. This brings us down to 5 possibilities each for u and v. Here is the list of possibilities for u:

[u1] $u \notin A$, $u \in B$ and $u \notin U$
[u2] $u \notin A$, $u \in B$ and $u \in U$
[u3] $u \in A$, $u \notin B$ and $u \notin U$
[u4] $u \in A$, $u \notin B$ and $u \in U$
[u5] $u \in A$, $u \in B$ and $u \in U$

Of course, there is a similar list for v, and the set of all 25 possibilities is given by taking one from each list.

Fortunately, separate arguments are not needed for each of the 25 cases. In particular, the four cases obtained by combining the first two possibilities for u and v involve situations where both points lie in B and therefore $|f(u) - f(v)| \leq K_2\,|u - v|$. Likewise, the nine cases obtained by combining the last three possibilities for u and v involve situations where both points lie in A and therefore $|f(u) - f(v)| \leq K_1\,|u - v|$.

This leaves us with twelve cases; six are given by taking one of the first two possibilities for u and one of the last three possibilities for v, and six are given by switching the roles of u and v. If we do the latter, we are down to six cases.

The cases $[u2] + [v2]$ and $[u2] + [v3]$ are situations where both point belong to U and therefore $|f(u) - f(v)| \leq K_0\,|u - v|$. In each of the remaining cases, at least one of u or v does not belong to U. Consider the continuous function on

$$\big((A \cup B) \times (A \cup B)\big) - U \times U$$

defined by the quotient

$$\frac{|f(u) - f(v)|}{|u - v|}\ .$$

Since the domain of this function is compact, it has a maximum value K_3, and thus we have $|f(u) - f(v)| \leq K_3\,|u - v|$ if (u, v) lies in the set described above.

If we take K to be the largest of the numbers K_i for $0 \leq i \leq 3$, then K will be a Lipschitz constant for f on $A \cup B$.■