SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 1

Fall 2014

I. Foundational material

I.1 : Basic set theory

Problems from Munkres, § 9, p. 64

2. $(a) - (c)$ For each of the first three parts, choose a 1–1 correspondence between the integers or the rationals and the positive integers, and consider the well-orderings that the latter inherit from these maps. For each nonempty subset, define the choice function to be the first element of that subset with respect to the given ordering.■

TECHNICAL FOOTNOTE. (*This uses material from a graduate level measure theory course.*) In Part (d) of the preceding problem, one cannot find a choice function without assuming something like the Axiom of Choice. The following explanation goes beyond the content of this course but is hopefully illuminating. The first step involves the results from Section I.3 which show that the set of all functions from $\{0,1\}$ to the nonnegative integers is in 1–1 correspondence with the real numbers. If one could construct a choice function over all nonempty subsets of the real numbers, then among other things one can prove that that there is a subset of the reals which is not Lebesgue measurable without using the Axiom of Choice (see any graduate level book on Lebesgue integration; for example, Section 3.4 of Royden). On the other hand, there are models for set theory in which every subset of the real numbers is Lebesgue measurable (see R. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Annals of Math. (2) **92** (1970), pp. 1–56). — It follows that one cannot expect to have a choice function for arbitrary families of nonempty subsets of the reals unless one makes some extra assumption related to the Axiom of Choice.

5. (a) For each $b \in B$ pick $h(b) \in A$ such that $f(a) = b$; we can find these elements by applying the Axiom of Choice to the family of subsets $f^{-1}[\{b\}]$ because surjectivity implies each of these subsets is nonempty. It follows immediately that $b = f(\,h(b)\,)$.■

(b) For each $x \in A$ define a function $g_x : B \to A$ whose graph consists of all points of the form

$$\big(\,f(a),\,a\,\big) \quad \in \quad B \times A$$

togther with all points of the form (b, x) if b does not lie in the image of A. The injectivity of f implies that this subset is the graph of some function g_x, and by construction we have $g_x \circ f(a) = a$ for all $a \in A$. This does NOT require the Axiom of Choice; for each $x \in A$ we have constructed an EXPLICIT left inverse to f. — On the other hand, if we had simply said that one should pick some element of A for each element of $B - f[A]$, then we WOULD have been using the Axiom of Choice.■

Additional exercise

1. The commutativity law for $\oplus$ holds because

$$B \oplus A = (B - A) \ \cup \ (A - B)$$

by definition and the commutativity of the set-theoretic union operation. The identity $A \oplus A = \emptyset$ follows because

$$A \oplus A = (A - A) \ \cup \ (A - A) = \emptyset$$

and $A \oplus \emptyset = A$ because

$$A \oplus \emptyset = (A - \emptyset) \ \cup \ (\emptyset - A) = A \ \cup \ \emptyset = A \ .$$

In order to handle the remaining associative and distributive identities it is necessary to write things out explicitly, using the fact that every Boolean expression involving a finite list of subsets can be written as a union of intersections of subsets from the list. It will be useful to introduce some algebraic notation in order to make the necessary manipulations more transparent. Let $X \supset A \cup B \cup C$ denote the complement of $Y \subset X$ by $\widehat{Y}$ (or by $Y^{\frown}$ if Y is some compound algebraic expression), and write $P \cap Q$ simply as PQ. Then the symmetric difference can be rewritten in the form $(A\widehat{B}) \cup (B\widehat{A})$. It then follows that

$$(A \oplus B) \oplus C = \left(A\widehat{B} \cup B\widehat{A}\right)\widehat{C} \, \cup \, C\left(A\widehat{B} \cup B\widehat{A}\right)^{\frown} =$$

$$A\widehat{B}\widehat{C} \cup B\widehat{A}\widehat{C} \cup C\left((\widehat{A} \cup B)(\widehat{B} \cup A)\right) = A\widehat{B}\widehat{C} \cup B\widehat{A}\widehat{C} \cup C\left(\widehat{A}\widehat{B} \cup AB\right) =$$

$$A\widehat{B}\widehat{C} \cup \widehat{A}B\widehat{C} \cup \widehat{A}\widehat{B}C \cup ABC \ .$$

Similarly, we have

$$A \oplus (B \oplus C) = A\left(B\widehat{C} \cup C\widehat{B}\right)^{\frown} \cup \left(B\widehat{C} \cup C\widehat{B}\right)\widehat{A} =$$

$$A\left((\widehat{B} \cup C)(\widehat{C} \cup B)\right) \cup B\widehat{C}\widehat{A} \cup C\widehat{B}\widehat{A} = A(\widehat{B}\widehat{C} \cup BC) \cup B\widehat{C}\widehat{A} \cup C\widehat{B}\widehat{A} =$$

$$A\widehat{B}\widehat{C} \cup \widehat{A}B\widehat{C} \cup \widehat{A}\widehat{B}C \cup ABC \ .$$

This proves the associativity of $\oplus$ because both expressions are equal to the last expression displayed above. The proof for distributivity is similar but shorter (the left side of the desired equation has only one $\oplus$ rather than two, and we only need to deal with monomials of degree 2 rather than 3):

$$A\,(B \oplus C) = A(B\widehat{C} \cup C\widehat{B}) = AB\widehat{C} \cup A\widehat{B}C$$

$$AB \oplus AC = (AB)(AC)^{\frown} \cup (AC)(AB)^{\frown} =$$

$$\left(AB(\widehat{A} \cup \widehat{C})\right) \cup \left(AC(\widehat{A} \cup \widehat{B})\right) = AB\widehat{C} \cup A\widehat{B}C$$

Thus we have shown that both of the terms in the distributive law are equal to the same set.∎

I.2 : Products, relations and functions

Problem from Munkres, § 4, p. 44

4. (*a*) This is outlined in the course notes.

Additional exercises

1. (*i*) Suppose that (x, y) lies in the left hand side. Then $x \in A$ and $y \in B \cap D$. Since the latter means $y \in B$ and $y \in D$, this means that

$$(x, y) \in (A \times B) \cap (A \times D) \ .$$

Now suppose that (x, y) lies in the set displayed on the previous line. Since $(x, y) \in A \times B$ we have $x \in A$ and $y \in B$, and similarly since $(x, y) \in A \times D$ we have $x \in A$ and $y \in D$. Therefore we have $x \in A$ and $y \in B \cap D$, so that $(x, y) \in A \times (B \cap D)$. Thus every member of the the first set is a member of the second set and vice versa, and therefore the two sets are equal.■

(*ii*) Suppose that (x, y) lies in the left hand side. Then $x \in A$ and $y \in B \cup D$. If $y \in B$ then $(x, y) \in A \times B$, and if $y \in D$ then $(x, y) \in A \times D$; in either case we have

$$(x, y) \in (A \times B) \cup (A \times D) \ .$$

Now suppose that (x, y) lies in the set displayed on the previous line. If $(x, y) \in A \times B$ then $x \in A$ and $y \in B$, while if $(x, y) \in A \times D$ then $x \in A$ and $y \in D$. In either case we have $x \in A$ and $y \in B \cup D$, so that $(x, y) \in A \times (B \cap D)$. Thus every member of the the first set is a member of the second set and vice versa, and therefore the two sets are equal.■

(*iii*) Suppose that (x, y) lies in the left hand side. Then $x \in A$ and $y \in Y - D$. Since $y \in Y$ we have $(x, y) \in A \times Y$, and since $y \notin D$ we have $(x, y) \notin A \times D$. Therefore we have

$$A \times (Y - D) \ \subset \ (A \times Y) - (A \times D) \ .$$

Suppose now that $(x, y) \in (A \times Y) - (A \times D)$. These imply that $x \in A$ and $y \in Y$ but $(x, y) \notin A \times D$; since $x \in A$ the latter can only be true if $y \notin D$. Therefore we have that $x \in A$ and $y \in Y - D$, so that

$$A \times (Y - D) \ \supset \ (A \times Y) - (A \times D) \ .$$

This proves that the two sets are equal.■

(*iv*) Suppose that (x, y) lies in the left hand side. Then we have $x \in A$ and $y \in B$, and we also have $x \in C$ and $y \in D$. The first and third of these imply $x \in A \cap C$, while the second and fourth imply $y \in B \cap D$. Therefore $(x, y) \in (A \cap C) \times (B \cap D)$ so that

$$(A \times B) \cap (C \times D) \ \subset \ (A \cap C) \times (B \cap D) \ .$$

Suppose now that (x, y) lies in the set on the right hand side of the displayed equation. Then $x \in A \cap C$ and $y \in B \cap D$. Since $x \in A$ and $y \in B$ we have $(x, y) \in A \times B$, and likewise since $x \in C$ and $y \in D$ we have $(x, y) \in C \times D$, so that

$$(A \times B) \cap (C \times D) \ \supset \ (A \cap C) \times (B \cap D) \ .$$

Therefore the two sets under consideration are equal.■

(v) Suppose that (x, y) lies in the left hand side. Then either we have $x \in A$ and $y \in B$, or else we have $x \in C$ and $y \in D$. The first and third of these imply $x \in A \cup C$, while the second and fourth imply $y \in B \cup D$. Therefore (x, y) is a member of $(A \cup C) \times (B \cup D)$ so that

$$(A \times B) \cup (C \times D) \ \subset \ (A \cup C) \times (B \cup D) \ .$$

Supplementary note: To see that the sets are not necessarily equal, consider what happens if $A \cap C = B \cap D = \emptyset$ but all of the four sets A, B, C, D are nonempty. Try drawing a picture in the plane to visualize this.■

(vi) Suppose that (x, y) lies in the left hand side. Then $x \in X$ and $y \in Y$ but $(x, y) \notin A \times B$. The latter means that the statement

$$x \in A \textbf{ and } y \in B$$

is false, which is logically equivalent to the statement

$$\textbf{either } x \notin A \textbf{ or } y \notin B \ .$$

If $x \notin A$, then it follows that $(x, y) \in \big((X - A) \times Y\big)$, while if $y \notin B$ then it follows that $(x, y) \in \big(X \times (Y - B)\big)$. Therefore we have

$$\big(X \times Y\big) - \big(A \times B\big) \ \subset \ \big(X \times (Y - B)\big) \cup \big((X - A) \times Y\big) \ .$$

Suppose now that (x, y) lies in the set on the right hand side of the containment relation on the displayed line. Then we have $(x, y) \in X \times Y$ and also

$$\textbf{either } x \notin A \textbf{ or } y \notin B \ .$$

The latter is logically equivalent to

$$x \in A \textbf{ and } y \in B$$

and this in turn means that $(x, y) \notin A \times B$ and hence proves the reverse inclusion of sets.■

2. Since the object of this exercise is to address ambiguities in notation, the key to solving this exercise is to find unambiguous ways of distinguishing between the two interpretations of the symbolism $f^{-1}[C]$. The formulation in the exercise goes part way towards doing this, but one must still be careful.

We shall systematically use the inverse function identities $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$ which relate a 1–1 onto function $f : A \to B$ with its inverse function $g : B \to A$. As in the statement of the exercise, let $C \subset B$.

Suppose that $a \in IMAGE\ g[C]$. Then $a = g(b)$ for some $b \in C$, which implies that $f(a) = f(g(b)) = b$, so that $f(a) \in C$ and hence $a \in INVERSE\ IMAGE\ f^{-1}[C]$. Conversely, suppose that $a \in INVERSE\ IMAGE\ f^{-1}[C]$. Then $f(a) \in C$, so that $g(f(a)) = a \in C$, which means that $a \in IMAGE\ g[C]$. Therefore each of the sets $IMAGE\ g[C]$ and $INVERSE\ IMAGE\ f^{-1}[C]$ is contained in the other, so the two sets are equal.■

3. We shall first prove that $\mathcal{R}^{\#}$ is an equivalence relation.

The relation is reflexive. The definition of $\mathcal{R}^{\#}$ stipulates that $x\ \mathcal{R}^{\#}\ x$.

The relation is symmetric. By the preceding step we need only consider the case where $x \neq y$. If we are given a finite sequence $\{v_0\ \cdots\ , v_m\}$ as described in the definition such that $v_0 = x$ and $v_m = y$, then the reverse sequence with $w_j = v_{m-j}$ satisfies the criterion in the definition implies with $w_0 = y$ and $w_m = x$.

The relation is transitive. Suppose that we have a finite sequence $\{v_0 \ \cdots \ , v_m\}$ as described in the definition such that $v_0 = x$ and $v_m = y$, and another finite sequence $\{u_0 \ \cdots \ , u_n\}$ as described in the definition such that $u_0 = y$ and $u_n = z$, then we can concatenate (string together) the original sequences into a new sequence $\{t_0 \ \cdots \ , t_{m+n}\}$ such that $t_j = v_j$ if $j \leq m$ and $t_j = u_{j-m}$ if $j \geq m$ (the formulas are consistent at the overlapping value m, for which $t_m = y$). This new sequence still satisfies the criterion in the definition with $t_0 = x$ and $t_{m+n} = z$.

To complete the proof, we need to verify that if $\mathcal{S}$ is an equivalence relation such that $x \ \mathcal{S} \ y$ whenever $x \ \mathcal{R} \ y$, then $u \ \mathcal{R}^{\#} \ v$ implies that $u \ \mathcal{S} \ v$. If $u = v$ then the latter follows because we are working with equivalence relations, which are reflexive. If $u \neq v$ and $u \ \mathcal{S} \ v$, let $\{t_0 \ \cdots \ , t_m\}$ be a sequence starting with u and ending with v such that the for each i we have either $t_i \ \mathcal{R} \ t_{i+1}$ or $t_{i+1} \ \mathcal{R} \ t_i$. Then by the hypotheses on $\mathcal{S}$ we know that $t_i \ \mathcal{S} \ t_{i+1}$ for all i, and therefore by repeated application of the transitivity property we have $u \ \mathcal{S} \ v$.■

4. We shall refer to the file `math205Asolutions01a.pdf` for drawings which may help explain the underlying ideas; as usual, the proof must be written so that it does not formally depend upon these drawings.

The first step is to show that if $(i, j) \in \mathcal{E}$, then every point of the form $(i + t, j + t)$ in B — where t runs through all admissible integers such that the point in question belongs to B — also lies in $\mathcal{E}$. In other words, if $i' - j' = i - j$, then $(i', j') \ \mathcal{E} \ (i, j)$. For points in B the difference values $i - j$ are the 15 integers between ± 7, so this shows that there are at most 15 equivalence classes (in the first drawing, the squares with $i - j$ = CONSTANT are on the diagonal lines and have the same color). To prove the assertion in the first sentence, observe that $(i, j) \ \mathcal{R} \ (i + \varepsilon, j + \varepsilon)$ for $\varepsilon = \pm 1$ by definition, and by definition of $\mathcal{E}$ this yields $(i, j) \ \mathcal{E} \ (i + \varepsilon, j + \varepsilon)$. The statement for general values of t now follows by repeated application of the final assertion in the previous sentence and the transitivity of $\mathcal{E}$.

Next, let $\mathcal{F}$ be the binary relation with $(i', j') \ \mathcal{F} \ (i, j)$ if $i' + j'$ and $i + j$ are both even or both odd. This is an equivalence relation by one of the exercises in Sutherland and the fact that two ordered pairs are $\mathcal{F}$ are related if and only if they have the same values under the function $\varphi : B \to \{\text{EVEN}, \text{ODD}\}$ whose value is determined by whether $i+j$ is even or odd. The definition of $\mathcal{R}$ implies that if $(i, j) \ \mathcal{R} \ (p, q)$ then both $i+j$ and $p+q$ are even or odd, and therefore $(i, j) \ \mathcal{F} \ (p, q)$ whenever $(i, j) \ \mathcal{R} \ (p, q)$. By Exercise 0, it follows that $(i, j) \ \mathcal{F} \ (p, q)$ whenever $(i, j) \ \mathcal{E} \ (p, q)$, and since $\mathcal{F}$ has two equivalence classes the equivalence relation $\mathcal{E}$ must also have at least two equivalence classes.

Finally, we need to show that $\mathcal{E}$ has exactly two equivalence classes. The idea is similar to that of the first step; namely, if $(i, j) \in \mathcal{E}$, then every point of the form $(i + t, j - t)$ in B — where t runs through all admissible integers such that the point in question belongs to B — also lies in $\mathcal{E}$. The main difference in the argument is the need to observe that we also have $(i, j) \ \mathcal{R} \ (i + \varepsilon, j - \varepsilon)$ for $\varepsilon = \pm 1$ by the definition of $\mathcal{R}$. By the same reasoning as in the first step, this implies that if $i' + j' = i + j$, then $(i', j') \ \mathcal{E} \ (i, j)$. — To conclude the argument, it suffices to observe that the set of all $(i, j) \in B$ with $i + j = 9$ the difference $i - j$ takes all odd values between -7 and $+7$, while the set of all (i, j) with $i + j = 8$ takes all even values between -6 and $+6$ (in the second drawing, observe how the two lines with slope -1 cut through all the lines with slope $+1$). This proves that there are at most two equivalence classes for $\mathcal{E}$, and by the preceding paragraph there must be precisely two equivalence classes.■

5. It turns out that, in order to make things less repetitive, the best place to start is by observing that if $[x] = [y]$ then $x \mathcal{S} y$. This follows from the reflexive property of the equivalence relation $\mathcal{R}_2$. Note that this also yields the reflexive property for $\mathcal{S}$.

Suppose now that $x \mathcal{S} y$, so that $[x] \mathcal{R}_2 [y]$. Since $\mathcal{R}_2$ is an equivalence relation, this means that $[y] \mathcal{R}_2 [x]$, which in turn implies that $y \mathcal{S} x$. Finally, suppose that $x \mathcal{S} y$ and $y \mathcal{S} z$, so that $[x] \mathcal{R}_2 [y]$ and $[y] \mathcal{R}_2 [z]$. Since $\mathcal{R}_2$ is transitive we have $[x] \mathcal{R}_2 [z]$, and this yields $x \mathcal{S} z$, so that $\mathcal{S}$ is an equivalence relation on X.■

I.3 : Cardinal numbers

Problem from Munkres, § 7, *p.* 51

4. (a) Let $\mathbb{Q}[t]$ denote the ring of polynomials with rational coefficients, and for each integer $d > 0$ let $\mathbb{Q}[t]_d$ denote the set of polynomials with degree equal to d. There is a natural identification of $\mathbb{Q}[t]_d$ with the subset of $\mathbb{Q}^{d+1}$ consisting of n-tuples whose last coordinate is nonzero, and therefore $\mathbb{Q}[t]_d$ is countable. Since a countable union of countable sets is countable (Munkres, Theorem 7.5, pp. 48–49), it follows that $\mathbb{Q}[t]$ is also countable.

Given an algebraic number α, there is a unique **monic** rational polynomial $p(t)$ of least (positive) degree such that $p(\alpha) = 0$ (the existence of a polynomial of least degree follows from the well-ordering of the positive integers, and one can find a monic polynomial using division by a positive constant; uniqueness follows because if p_1 and p_2 both satisfy the condition then $p_1 - p_2$ is either zero or a polynomial of lower degree which has α as a root). Let p_α be the polynomial associated to α in this fashion. Then p may be viewed as a function from the set $\mathcal{A}$ of algebraic numbers into $\mathbb{Q}[t]$; if f is an arbitrary element of degree $d \geq 0$, then we know that there are at most d elements of $\mathcal{A}$ that can map to p (and if $f = 0$ the inverse image of $\{f\}$ is empty). Letting $\mathcal{A}_f$ be the inverse image of f, we see that $\mathcal{A} = \cup_f \mathcal{A}_f$, so that the left hand side is a countable union of finite sets and therefore is countable.■

(b) Since every real number is either algebraic or transcendental but not both, we clearly have

$$2^{\aleph_0} = |\mathbb{R}| = |\text{algebraic}| + |\text{transcendental}| \;.$$

We know that the algebraic numbers are countable, so if the transcendental numbers are also countable the right hand side of this equation reduces to $\aleph_0 + \aleph_0$, which is equal to $\aleph_0$, a contradiction. Therefore the set of transcendental numbers is uncountable (in fact, its cardinality is $2^{\aleph_0}$ but the problem did not ask for us to go any further).■

Problem from Munkres, § 9, *p.* 62

5. (a) For each $b \in B$ let $L_b \subset A$ be the inverse image $f^{-1}(\{b\})$. Using the axiom of choice we can find a function g that assigns to each set L_b a point $g^*(L_b) \in L_b$. Define $g(b) = g^*(L_b)$; by construction we have that $g(b) \in f^{-1}(\{b\})$ so that $f(g(b)) = b$. This means that $f \circ g = \text{id}_B$ and that g is a right inverse to f.■

(b) Given an element $z \in A$ define a map $g_z : B \to A$ as follows: If $b = f(a)$ for some a let $g_z(b) = a$. This definition is unambiguous because there is at most one $a \in A$ such that $f(a) = b$. If b does not lie in the image of f, set $g_z(b) = z$. By definition we then have $g_z(f(a)) = a$ for all $a \in A$, so that $g_z \circ f = \text{id}_A$ and g_z is a left inverse to f. Did this use the Axiom of Choice? No.

What we actually showed was that for each point of A there is an associated left inverse. However, if we had simply said, "pick some point $z_0 \in A$ and define g using z_0," then we would have used the Axiom of Choice.■

Problem from Munkres, § 11, p. 72

8. (a) Note first that $\beta \notin A$ for otherwise it would be a linear combination of elements in A for trivial reasons.

Suppose the set in question is not linearly independent; then some finite subset C is not linearly independent, and we may as well add β to that subset. It follows that there is a relation

$$x_\beta \beta + \sum_{\gamma \in A \cap C} x_\gamma \gamma = 0$$

where not all of the coefficients x_β or x_γ are equal to zero. In fact, we must have $x_\beta \neq 0$ for otherwise there would be some nontrivial linear dependence relationship in $A \cap C$, contradicting our original assumption on A. However, if $x_\beta \neq 0$ then we can solve for β to express it as a linear combination of the vectors in $A \cap C$, and this contradicts our assumption on β. Therefore the set in question must be linearly independent.■

(b) Let $\mathbf{X}$ be the partially ordered set of linearly independent subsets of V, with inclusion as the partial ordering. In order to apply Zorn's Lemma we need to know that an arbitrary linearly ordered subset $\mathbf{L} \subset \mathbf{X}$ has an upper bound in in $\mathbf{X}$. Suppose that $\mathbf{L}$ consists of the subsets A_t; it will suffice to show that the union $A = \cup_t A_t$ is linearly independent, for then A will be the desired upper bound.

We need to show that if C is a finite subset of A then C is linearly independent. Since each A_t is linearly independent, it suffices to show that there is some r such that $C \subset A_r$, and we do this by induction on $|C|$. If $|C| = 1$ this is clear because $\alpha \in A$ implies $\alpha \in A_t$ for some t. Suppose we know the result when $|C| = k$, and let $D \subset A$ satisfy $|D| = k + 1$. Write $D = D_0 \cup \gamma$ where $\gamma \notin D_0$. Then there is some u such that $D_0 \subset A_u$ and some v such that $\gamma \in A_v$. Since $\mathbf{L}$ is linearly ordered we know that either $A_u \subset A_v$ or vice versa; in either case we know that D is contained in one of the sets A_u or A_v. This completes the inductive step, which in turn implies that A is linearly independent and we can apply Zorn's Lemma.■(c) Let A be a maximal element of $\mathbf{X}$ whose existence was guaranteed by the preceding step in this exercise. We claim that every vector in V is a linear combination of vectors in A. If this were not the case and β was a vector that could not be expressed in this fashion, then by the first step of the exercise the set $A \cup \{\beta\}$ would be linearly independent, contradicting the maximality of A.■

Additional exercises

1. Let X be the set in question, and let $Y \subset X$ be the subset of all one point subsets. Since there is a 1–1 correspondence between $\mathbb{R}$ and Y it follows that $2^{\aleph_0} = |\mathbb{R}| = |Y| \leq |X|$. Now write X as a union of subfamilies X_n where $0 \leq n \leq \infty$ such that the cardinality of every set in X_n is n if $n < \infty$ and the cardinality of every set in X_∞ is $\aleph_0$.

Suppose now that $n < \infty$. Then X_n is in 1–1 correspondence with the set of all points $(x_1, \cdots, x_n)$ in $\mathbb{R}^n$ such that $x_1 < \cdots < x_n$ (we are simply putting the points of the subset in order). Therefore $|X_0| = 1$ and $|X_n| \leq 2^{\aleph_0}$ for $1 \leq n < \infty$, and it follows that $\cup_{n<\infty} X_n$ has cardinality at most

$$\aleph_0 \cdot 2^{\aleph_0} \ \leq \ 2^{\aleph_0} \cdot 2^{\aleph_0} \ = \ 2^{\aleph_0} \ .$$

So what can we say about the cardinality of X_∞? Let S be the set of all infinite sequences in $\mathbb{R}$ indexed by the positive integers. For each choice of a 1–1 correspondence between an element of X_∞ and $\mathbb{N}^+$ we obtain an element of S, and if we choose one correspondence for each element we obtain a 1–1 map from X_∞ into S. By definition $|S|$ is equal to $\left(2^{\aleph_0}\right)^{\aleph_0}$, which in turn is equal to $2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}$; therefore we have $|X_\infty| \leq 2^{\aleph_0}$. Putting everything together we have

$$|X| = |\cup_{n<\infty} \;\; X_n| + |X_\infty| \leq 2^{\aleph_0} + 2^{\aleph_0} = 2^{\aleph_0}$$

and since we have already established the reverse inequality it follows that $|X| = 2^{\aleph_0}$ as claimed.■

IMPORTANT FOOTNOTE. The preceding exercise relies on the generalization of the law of exponents for cardinal numbers

$$\gamma^{\alpha\beta} = \left(\gamma^\alpha\right)^\beta$$

that was stated at the end of Section I.3 of the course notes without proof. For the sake of completeness we shall include a proof.

Choose sets A, B, C so that $|A| = \beta$, $|B| = \alpha$ (note the switch!!) and $|C| = \gamma$, and let $\mathbf{F}(S,T)$ be the set of all (set-theoretic) functions from one set S to another set T. With this terminology the proof of the cardinal number equation reduces to finding a 1–1 correspondence

$$\mathbf{F}(A \times B, C) \longleftrightarrow \mathbf{F}\big(A, \; \mathbf{F}(B,C) \;\big) \; .$$

In other words, we need to construct a 1–1 correspondence between functions $A \times B \to C$ and functions $A \to \mathbf{F}(B,C)$. In the language of category theory this is an example of an *adjoint functor* relationship.

Given $f : A \times B \to C$, construct $f^* : A \to \mathbf{F}(B,C)$ by defining $f^*(a) : B \to C$ using the formula

$$\left[f^*(a)\right](b) = f(a,b) \; .$$

This construction is onto, for if we are given $h^* : A \to \mathbf{F}(B,C)$ and we define $f : A \times B \to C$ by the formula

$$f(a,b) = \left[h(a)\right](b)$$

then $f^* = h$ by construction; in detail, one needs to check that $f^*(a) = h(a)$ for all $a \in A$, which amounts to checking that $[f^*(a)](b) = [h(a)](b)$ for all a and b — but both sides of this equation are equal to $f(a,b)$. To see that the construction is 1–1, note that $f^* = g^* \iff f^*(a) = g*^(a)$ for all a, which is equivalent to $[f^*(a)](b) = [g^*(a)](b)$ for all a and b, which in turn is equivalent to $f(a,b) = g(a,b)$ for all a and b, which is equivalent to $f = g$. Therefore the construction sending f to f^* is 1–1 onto as required.■

For the record, the other exponential law

$$\left(\beta \cdot \gamma\right)^\alpha = \beta^\alpha \cdot \gamma^\alpha$$

may be verified by first noting that it reduces to finding a 1–1 correspondence between $\mathbf{F}(A, B \times C)$ and

$$\mathbf{F}(A,B) \times \mathbf{F}(A,C) \; .$$

This simply reflects the fact that a function $f : A \to B \times C$ is completely determined by the ordered pair of functions $p_B \circ f$ and $p_C \circ f$ where p_B and p_C are the coordinate projections from $B \times C$ to B and C respectively.■

2. The inequality means that there is a 1–1 mapping j from some set A_0 with cardinality α to a set B with cardinality β. Since the cardinality of X equals β it follows that there is a 1–1 correspondence $f : B \to X$. If we take $A = j(f(A_)))$, then $A \subset X$ and $|A| = \alpha$.■

3. We shall prove this using Zorn's Lemma (also known as the Kuratowski-Zorn Lemma) and the partial ordering of $\mathcal{F}$ by inclusion. It suffices to show that linearly ordered subfamilies of $\mathcal{F}$ have upper bounds in $\mathcal{F}$.

Suppose that $\mathcal{L} \subset \mathcal{F}$ is linearly ordered. Define

$$L^* \quad = \quad \bigcup_{L_\alpha \in \mathcal{L}} L_\alpha .$$

Clearly $L^* \supset L_\alpha$ for all α, so it is only necessary to prove that $L^* \in \mathcal{F}$. Suppose that $A = \{a_1, \ \cdots \ , a_k\}$ is a finite subset of L^*, and for $1 \leq j \leq k$ choose $\alpha(j)$ such that $a_j \in L_{\alpha(j)}$. Since $\mathcal{L}$ is linearly ordered there is some m such that $1 \leq m \leq k$ and $L_{\alpha(j)} \subset L_{\alpha(m)}$ for all m. The latter implies that $A \subset L_{\alpha(m)}$. Since $L_{\alpha(m)} \in \mathcal{F}$ the first defining property of finite character implies that $A \in \mathcal{F}$, and since $\mathcal{F}$ has finite character the second defining property implies that $L^* \in \mathcal{F}$, which is what we wanted to prove.■

I.4 : The real number system

Problem from Munkres, § 4, p. 35

9. (c) We shall follow the hint. Part (b) states that if a real number x is not an integer, then there is a unique integer n such that $n < x < n + 1$.

The solution to (c) has two cases depending upon whether or not y is an integer. If $y \in \mathbb{Z}$, then the same is true for $y + 1$ and we have

$$y \quad < \quad y \ + \ 1 \quad < \quad y \ + \ (y - x) \quad = \quad x$$

so that $y + 1$ is an integer with the required properties. Suppose now that $y \notin \mathbb{Z}$, so that there is a unique integer m such that $m < y < m + 1$. We then have

$$x \quad = \quad y \ + \ (x - y) \quad m \ + \ 1$$

so that $y < m + 1 < x$.■

Remark. Of course, the integer n in the conclusion of the preceding exercise is not necessarily unique; for example if we have the stronger inequality $x - y \geq 2$ then there are at least two integers between x and y.

Additional exercise

1. The answer is emphatically **NO,** and there are many counterexamples. Let $\{ x_\alpha \}$ be a basis for $\mathbb{R}$ as a vector space over $\mathbb{Q}$. If $f : \mathbb{R} \to \mathbb{R}$ is a $\mathbb{Q}$-linear map, then f satisfies the condition in the problem. Thus it is only necessary to find examples of such maps that are not multiplication by a constant. Since $\mathbb{R}$ is uncountable, a basis for it over the rationals contains infinitely many elements. Pick one element x_0 in the basis, and consider the unique $\mathbb{Q}$-linear transformation f which sends

x_0 to itself and all other basis vectors to zero. Then f is nonzero but is neither 1–1 nor onto. In contrast, a mapping of the form $T_c(x) = c \cdot x$ for some fixed real number c is 1–1 and onto if $c \neq 0$ and zero if $c = 0$. Therefore there is no c such that $f = T_c$.■

Generalization – more difficult to verify. In fact, the cardinality of the set of all mappings f with the given properties is $2^{\mathbf{c}}$, where as usual $\mathbf{c} = 2^{\aleph_0}$ (the same as the cardinality of all maps from $\mathbb{R}$ to itself). To see this, first note that the statement in parentheses follows because the cardinality of the set of all such mappings is

$$\mathbf{c}^{\mathbf{c}} \quad = \quad \left(2^{\aleph_0}\right)^{\mathbf{c}} \quad = \quad 2^{\aleph_0 \cdot \mathbf{c}} \quad = \quad 2^{\mathbf{c}} \ .$$

To prove the converse, we first claim that a basis for $\mathbb{R}$ over $\mathbb{Q}$ must contain $\mathbf{c}$ elements. Note that the definition of vector space basis implies that $\mathbb{R}$ is in 1–1 correspondence with the set of finitely supported functions from B to $\mathbb{Q}$, where B is a basis for $\mathbb{R}$ over $\mathbb{Q}$ and finite support means that the coordinate functions are nonzero for all but finitely many basis elements. Thus if β is the cardinality of B, then the means that the cardinality of $\mathbb{R}$ is β (the details of checking this are left to the reader). For each subset A of B we may define a $\mathbb{Q}$-linear map from $\mathbb{R}$ to itself which sends elements of B to themselves and elements of the difference set $B - A$ to zero. Different subsets determine different mappings, so this shows that the set of all f satisfying the given condition has cardinality at least $2^{\mathbf{c}}$. Since the first part of the proof shows that the cardinality is at most $2^{\mathbf{c}}$, this completes the argument.■

Remark. By the results of Unit II, if one also assumes that the function f is **continuous,** then the answer becomes **YES.** One can use the material from Unit II to prove this quickly as follows: If $r = f(1)$, then by induction and $f(-x) = -f(x)$ implies that $f(a) = r \cdot a$ for all $a \in \mathbb{Z}$; if $a = p/q \in \mathbb{Q}$, then we have

$$q \cdot f(a) \quad = \quad f(q \cdot a) \quad = \quad f(p) \quad = \quad r\,p\ , \qquad \text{yielding} \qquad f(a) \quad = \quad r \cdot \frac{p}{q}$$

so that $f(a) = r \cdot a$ for all $a \in \mathbb{Q}$. By the results of Section II.4, if f and g are two continuous functions such that $f(a) = g(a)$ for all rational numbers a, then $f = g$. Taking g to be multiplication by r, we conclude that f must also be multiplication by c.■

2. Follow the hint. If A has an upper bound and β is given as in the hint, then by construction β is an upper bound for A, and it is the least such upper bound.■

3. Since $n(k)$ is strictly increasing we must have $n(k) \geq k$ for all k (Proof by induction: $n(0) \geq 0$, and if $n(m) \geq n$ then $n(m+1) >\geq n(m) + 1 \geq m + 1$ because $n(k)$ is strictly increasing). By the definition of a limit for a sequence, for every $\varepsilon > 0$ there is a positive integer N such that $n \geq N$ implies $|a_n - L| < \varepsilon$. Therefore if $k \geq N$ then $n(k) \geq n$, so that $|a_{n(k)} - L| < \varepsilon$.■

NOTE. The contrapositive forms of this result are particularly useful for showing that a specific sequence has no limit. If either (a) there is a subsequence with no limit, or (b) there are two subsequences with different limits, then the original sequence cannot have a limit.■

4. (i) The sequence of left hand endpoints $\{a_n\}$ is bounded from above by b_k, where k is an arbitrary nonnegative integer, while the sequence of right hand endpoints $\{b_n\}$ is bounded from below by a_k, where k is an arbitrary nonnegative integer. Therefore the set of left hand endpoints has a least upper bound A, and $A \leq b_k$ for all k. Similarly, the set of right hand endpoints has a greatest lower bound B, and by the preceding sentence we must have $A \leq B$. Therefore every point p satisfying $A \leq p \leq B$ must lie in each of the intervals. — Note that $A = B$ is possible; for example, consider the intervals $\left[2^{-k}, 2^k\right]$.

(ii) The answer is definitely **NO**. We shall give a counterexample involving $\sqrt{2}$. Since there is always a rational number between two real numbers, for each n we can find rational number a_n and b_n such that

$$\sqrt{2} \; - \; \left(\tfrac{1}{2}\right)^n \;\; < \;\; a_n \;\; < \;\; \sqrt{2} \;\; < \;\; b_n \;\; < \;\; \sqrt{2} \; + \; \left(\tfrac{1}{2}\right)^n \; .$$

It follows that if $p \in [a_n, b_n]$ then $|\sqrt{2} - p| \leq \left(\frac{1}{2}\right)^n$, so if p lies in each interval then this inequality holds for every n. But the latter implies that $|p - \sqrt{2}| = 0$, so that $p = \sqrt{2}$ and the latter is the only point which lies on each interval. Since $\sqrt{2}$ is irrational, there is no rational number which lies on all of the intervals.■

II. Metric and topological spaces

II.1 : Metrics and topologies

Problem from Munkres, § 13, *p.* 83

3. X lies in the family because $X - X = \emptyset$ and the latter is finite, while $\emptyset$ lies in the family because $X - \emptyset = X$. Suppose U_α lies in the family for all $\alpha \in A$. To determine whether their union lies in the family we need to consider the complement of that union, which is

$$X - \bigcup_\alpha U_\alpha \quad = \quad \bigcap_\alpha X - U_\alpha \ .$$

Each of the sets in the intersection on the right hand side is either countable or all of X. If at least one of the sets is countable then the whole intersection is countable, and the only other alternative is if each set is all of X, in which case the intersection is X. In either case the complement satisfies the condition needed for the union to belong to $\mathbf{T}_c$. Suppose now that we have two sets U_1 and U_2 in the family. To decide whether their intersection lies in the family we must again consider the complement of $U_1 \cap U_2$, which is

$$(X - U_1) \cup (X - U_2) \ .$$

If one of the two complements in the union is equal to X, then the union itself is equal to X, while if neither is equal to X then both are countable and hence their union is countable. In either case the complement of $U_1 \cap U_2$ satisfies one of the conditions under which $U_1 \cap U_2$ belongs to $\mathbf{T}_c$.

What about the other family? Certainly $\emptyset$ and X belong to it. What about unions? Suppose that X is an infinite set and that U and V lie in this family. Write $E = X - U$ and $F = X - V$; by assumption each of these subsets is either infinite or empty. Is the same true for their intersection? Of course not! Take X to be the positive integers, let E be all the even numbers and let F be all the prime numbers. Then E and F are infinite but the only number they have in common is 2. — Therefore the family $\mathbf{T}_\infty$ is not necessarily closed under unions and hence it does not necessarily define a topology for X.■

Problems from Munkres, § 16, *pp.* 91 – 92

1. We are given $A \subset Y \subset X$. Given $P \subset Q$ and a topology $\mathbf{T}$ on P, then $\mathbf{T}|Q$ will denote the subspace topology on Q, and with this notation the conclusion to be verified is that $\mathbf{T}|A = (\mathbf{T}|Y)\,|A$.

If U is open in $\mathbf{T}|A$, then $U = W \cap A$ where W is open in X. Since $A \subset Y$, we have $A = Y \cap A$ and hence $U = W \cap Y \cap A$, where $W \cap Y$ is open in Y by definition of the subspace topology. Therefore we have $\mathbf{T}|A \subset (\mathbf{T}|Y)\,|A$.

Conversely, a subset in $(\mathbf{T}|Y)\,|A$ has the form $V \cap A$ where V is open in Y. The latter condition translates into the statement $V = W \cap Y$ where W is open in X. Therefore we have $U = V \cap A = W \cap Y \cap A$ where W is open in X, and since $Y \cap A = A$ the latter means $U = W \cap A$ where W is open in X. Therefore we also have $\mathbf{T}|A \supset (\mathbf{T}|Y)\,|A$.

Finally, since each topology contains the other, they are equal.■**3.** A is open in both Y and $\mathbb{R}$. B is open in Y but not in $\mathbb{R}$. Neither C not D is open in either of Y or $\mathbb{R}$. E is the union of the open intervals

$$\left(\frac{1}{n+1}, \frac{1}{n}\right)$$

where n runs over all positive integers; this set is open in both Y and $\mathbb{R}$.■

0. (*i*) If X is a finite set with n elements there are 2^n subsets of X and hence 2^{2^n} families of subsets of X. Since a topology on X is a family of subsets, this means there are at most 2^{2^n} topologies on X.■

(*ii*) Since the empty set and $\{0,1\}$ are in every topology $\mathbf{T}$ on $\{0,1\}$, the problem reduces to determining which subsets with one element belong to $\mathbf{T}$. The extreme cases — no such subsets in $\mathbf{T}$ or both $\{0\}$ and $\{1\}$ in $\mathbf{T}$ — correspond to the indiscrete and discete topologies respectively. There are two remaining posssibilities in which exactly one of $\{0\}$ and $\{1\}$ belongs to $\mathbf{T}$. Since we are dealing with finite sets, the test for a topology is whether the family of nonempty proper subsets is closed under twofold unions and intersections (twofold + induction $\Leftarrow$ finite unions and intersections, and every union of subsets must be a finite union because there are only finitely many subsets). But each of the families whose nonempty proper subsets are just $\{0\}$ and $\{1\}$ is closed under union and intersection ($A \cup A = A = A \cap A$), so this means that there are FOUR topologies — the discrete and indiscrete topologies, and two more topologies which contain exactly one subset with exactly one element (these are sometimes called *Sierpiński spaces*).■

Note. Clearly one can ask similar questions about topologies on other finite sets of the form $\{0, 1, \cdots, n\}$, but things quickly become fairly complicated. The case $n = 2$ is worked out completely in the following online document:

`http://fdslive.oup.com/www.oup.com/booksites/pdf/uk/companion/9780199563081/S.7.pdf`

It turns out that there are 29 different topologies on $\{0, 1, 2\}$.

1. In the definition of $\mathbf{d}_p$ we tacitly assume that $a \neq b$ and set $\mathbf{d}_p(a,a) = 0$ for all a. The nonnegativity of the function and its vanishing if and only if both variables are equal follow from the construction, as does the symmetry property $\mathbf{d}_p(a,b) = \mathbf{d}_p(b,a)$. The Triangle Inequality takes more insight. There is a special class of metric spaces known as **ultrametric spaces**, for which

$$\mathbf{d}(x,y) \leq \max\{\mathbf{d}(x,z), \mathbf{d}(y,z)\}$$

for all $x, y, z \in X$; the Triangle Inequality is an immediate consequence of this ultrametric inequality.

To establish this for the metric $\mathbf{d}_p$, we may as well assume that $x \neq y$ because if $x = y$ the ultrametric inequality is trivial (the left side is zero and the right is nonnegative). Likewise, we may as well assume that all three of x, y, z are distinct, for otherwise the ultrametric inequality is again a triviality. But suppose that $\mathbf{d}_p(x,y) = p^{-r}$ for some nonnegative integer r. This means that $x - y = p^r q$ where q is not divisible by p. If the ultrametric inequality is false, then p^{-r} is greater than either of either $\mathbf{d}_p(x,z)$ and $\mathbf{d}(y,z)$, which in turn implies that both $x - z$ and $y - z$ are divisible by p^{r+1}. But these two conditions imply that $x - y$ is also divisible by p^{r+1}, which is a contradiction. Therefore the ultrametric inequality holds for $\mathbf{d}_p$.

One curious property of this metric is that it takes only a highly restricted set of values; namely 0 and all fractions of the form p^{-r} where r is a nonnegative integer.■

2. Since U is open in A there is an open subset W in X such that $U = W \cap A$, and since $U \subset V$ we even have $U = V \cap U = V \cap W \cap A$. But $V \cap W$ is contained in the union of $U = V \cap W \cap A$ and $V - A$, and thus we have

$$U \cup (V - A) \;\subset\; (V \cap W) \cup (V - A) \;\subset\; (U \cup (V - A)) \cup (V - A) \;\subset\; U \cup (V - A)$$

so that $U \cup (V - A) = (V \cap W) \cup (V - A)$. Since A is closed the set $V - A$ is open, and therefore the set on the right hand side of the preceding equation is also open; of course, this means that the set on the left hand side of the equation is open as well.■

3. ($\Longrightarrow$) If $A = E$ then E is open in itself, and therefore the first condition implies that E is open in X. ($\Longleftarrow$) If E is any subset of X and A is open in E then $A = U \cap E$ where U is open in X. But we also know that E is open in X, and therefore $A = U \cap E$ is also open in X.■

4. (i) Suppose that $u, v \in N_\varepsilon(x)$. Then $\mathbf{d}(u,x) < \varepsilon$ and $\mathbf{d}(v,x) < \varepsilon$, so by the Triangle Inequality we have $\mathbf{d}(u,v) \leq \mathbf{d}(u,x) + \mathbf{d}(v,x) < \varepsilon + \varepsilon = 2\,\varepsilon$.■

(ii) Suppose that $\mathbf{d}(a_1,a_2) \leq M$ for $a_1, a_2 \in A$, and suppose also that $B \subset A$. Then $b_1, b_2 \in B$ implies $b_1, b_2 \in A$ and therefore $\mathbf{d}(b_1,b_2) \leq M$, which means that B is also bounded.

Suppose now that A and B are bounded, with $\mathbf{d}(a_1,a_2) \leq K$ for $a_1, a_2 \in A$ and $\mathbf{d}(b_1,b_2) \leq L$ for $b_1, b_2 \in B$. If $u, v \in A \cup B$ there are essentially three possibilities:

$$u, v \ \in \ A \ , \qquad u, v \ \in \ B \ , \qquad u \ \in \ A \quad \text{and} \quad v \ \in \ B$$

(Strictly speaking, $u \in B$ and $v \in A$ is also possible, but it can be handled by reversing the roles of u and v in the third displayed case.) In the first two cases we have $\mathbf{d}(u,v) \leq K$ and $\mathbf{d}(u,v) \leq L$ respectively. In the third case, let $a_0 \in A$ and $b_0 \in B$. Then two applications of the Triangle Inequality imply that

$$\mathbf{d}(u,v) \ \leq \ \mathbf{d}(u,a_0) \ + \ \mathbf{d}(a_0,b_0) \ + \ \mathbf{d}(b-0,v) \ \leq \ K \ + \ \mathbf{d}(a_0,b_0) \ + \ L$$

and therefore the distance between two points in $A \cup B$ is bounded by the right hand side of the display in all cases.

An infinite union of bounded subsets can be unbounded. One of the simplest examples is given by $U_n = (-n,n) \subset \mathbb{R}$. By (i) we know that the distance between two points of U_n is bounded from above by $2n$, but the union of all the sets U_n is the entire real line, which is unbounded.

There are also plenty of other families $\{A_n\}$ such that each A_n is bounded but $\cup_n\ A_n$ is not, and it is even possible to find examples where there is a single constant M such that for each n we have $\mathbf{d}(u,v) \leq M$ for all $u, v \in A_n$. A specific example along these lines is $A_n = [n, n+1]$ for n a nonnegative integer. For this family one can take $M = 1$ for all n, but $\cup_n\ A_n \ = \ [0,\infty)$, which is unbounded.■

5. (i) Since $\mathbb{R} \cap A = A$, $\mathbb{R} \cup A = \mathbb{R}$, $\emptyset \cap A = \emptyset$ and $\emptyset \cup A = A$ for all subsets $A \subset \mathbb{R}$, we only need to show that an arbitrary union or twofold intersection of nonempty proper subsets in $\mathbf{U}$ is also in $\mathbf{U}$ (in which case the intersection might be empty or the union might be all of $\mathbb{R}$).

Since the sets (b,∞) are open in the metric topology, it follows that $\mathbf{U}$ is contained in the metric topology. The containment is proper because nonempty open subsets in $\mathbf{U}$ never have an upper bound in $\mathbb{R}$ and there are many metrically open subsets that do — for example, the interval $(-1,1)$.

Twofold intersections. Given (b_1,∞) and (b_2,∞), their intersection is (c,∞), where c is the larger of b_1 and b_2.

Arbitrary unions. Suppose that we are given nonempty proper open subsets (b_α,∞) for $\alpha \in A$. We claim that

$$\bigcup_{\alpha \in A} (b_\alpha,\infty) \quad = \quad (b^*,\infty)$$

where b^* is either $-\infty$ or the greatest lower bound of $B = \{\ b_\alpha \mid \alpha \in A\}$. Since $b^* \leq b_\alpha$ for all α it is clear that the left hand side is contained in the right hand side. It remains to prove the reverse inclusion.

Unbounded case. If B is unbounded, then for each $x \in \mathbb{R}$ there is some b_γ such that $b_\gamma < x$, and therefore $x \in (b_t, \infty)$ and hence x lies in the union.

Bounded case. Suppose now that B is bounded, and let b^* be its greatest lower bound. If $x > b^*$, then x is not a lower bound for B, so there is some b_γ such that $b_\gamma < x$. We then have $x \in (b_\gamma, \infty)$ and hence once again x lies in the union.■

(ii) This is very similar to (i), and the simplest way to dispose of it is in two steps:

(a) If $(X, \mathbf{T})$ is a topological space and $f : X \to Y$ is 1–1 onto, then the family $f_*\mathbf{T}$ of all sets of the form $f[V]$, where V is open in X, is a topology on Y.

(b) If $\mathbf{U}$ is the topology in (i) and $g(x) = -x$ on $\mathbb{R}$, then $g_*\mathbf{U} = \mathbf{L}$.

The second statement is clear because g maps (b, ∞) to $(-\infty, -b)$ and vice versa. The first statement follows because $g[\emptyset] = \emptyset$, $g[X] = Y$, $g[U_1 \cap U_2] = g[U_1] \cap g[U_2]$ and $g[\cup_\alpha U_\alpha] = \cup_\alpha g[U_\alpha]$ (the first and last hold for arbitrary maps, while the middle two hold if g is a 1–1 onto mapping).■

(iii) The nonempty proper subsets in $\mathbf{U} \cup \mathbf{L}$ either do not have an upper bound or do not have a lower bound. Each set $(x - \varepsilon, x + \varepsilon)$ is an intersection of one set in $\mathbf{L}$ and one set in $\mathbf{U}$, and if $\mathbf{T}$ is a topology containing $\mathbf{U} \cup \mathbf{L}$ then this intersection must lie in $\mathbf{T}$. Since the constructed interval has both upper and lower bounds, it cannot belong to $\mathbf{U} \cup \mathbf{L}$, and therefore the latter cannot be a topology on $\mathbb{R}$.■

6. We shall first verify that if X is a topological space with base $\mathcal{B}$, then the subfamilies $\mathcal{B}_x$ satisfy the given conditions.

Verification of **(N1)**. By definition, if $u \in \mathcal{B}_x$ then $x \in U$.

Verification of **(N2)**. The set $U_1 \cap U_2$ is an open subset containing x. Since $\mathcal{B}$ is a base for X we know that $U_1 \cap U_2$ is a union of open subsets in $\mathcal{B}$; express $U_1 \cap U_2$ as such a union V_α, and choose V^* such that $x \in V^*$.

Verification of **(N3)**. Let $W = U$; by definition, this set belongs to $\mathcal{B}_y$.

Verification of **(N4)**. If the condition holds, then we know that each V_x is open, and we also know that

$$U \;=\; \bigcup_{x \in U} \{x\} \;\subset\; \bigcup_{x \in U} V_x \;\subset\; U$$

which implies that the two subsets in the middle are equal to U. The third set in the sequence is open because each V_x is open, and therefore U itself must be open.

We shall now prove that if we are given the data $\mathcal{B}_x$ for each $x \in X$ such that **(N1)**–**(N3)** are satisfied, then the union $\cup_{x \in X} \mathcal{B}_x$ is the base for some topology on the set X; by the definition of a base for a topology, property (**N4**) will also hold in this case.

Let $\mathbf{T}$ be the topology generated by $\mathcal{U} = \cup_{x \in X} \mathcal{B}_x$; it will suffice to show that the latter is in fact a basis for $\mathbf{T}$. This will be true if each finite intersection of sets in $\mathcal{U}$ can be expressed as a union of subsets in $\mathcal{U}$. By induction and the distributivity properties of unions and intersections, this reduces to verifying the assertion in the preceding sentence for twofold intersections (because a threefold intersection is a union of twofold intersections, *etc.* — fill in the details!). Let $p, q \in X$, suppose that we are given open subsets $V_p \in \mathcal{B}_p$ and $V_q \in \mathcal{B}_q$, and suppose that $z \in V_p \cap V_q$. By **(N3)** there are sets $W_p, W_q \in \mathcal{B}_z$ such that $z \in W_p \subset V_p$ and $z \in W_q \subset V_q$. Therefore by **(N2)** there is some subset $W_z^* \subset W_p \cap W_q$ in $\mathcal{B}_z$. It follows that $W_z^* \subset W_p \cap W_q$ and that $V_p \cap V_q$ is the union of such sets W_z^*; therefore the intersection of two sets in $\mathcal{U}$ is a union of subsets in $\mathcal{U}$, and as noted before this implies that $\mathcal{U}$ is a base for $\mathbf{T}$.■

NOTE. If X is a metric space, the for each $x \in X$ the families of subsets $\{\ N_{1/n}(x) \mid n \in \mathbb{N}^+\ \}$ satisfy conditions **(N1)** – **(N4)**. The family $\mathcal{B}_x$ is called an *open neighborhood base at the point*

$x \in X$. — In this terminology, we can say that a topological space is completely determined by describing abstract open neighborhood bases at all the points of X.