SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 3

Fall 2014

III. Spaces with special properties

III.1 : Compact spaces – I

Problems from Munkres, § 26, *pp.* 170 – 172

3. Suppose that $A_i \subset X$ is compact for $1 \leq i \leq n$, and suppose that $\mathcal{U}$ is a family of open subsets of X whose union contains $\cup_i A_i$. Then for each i there is a finite subfamily $\mathcal{U}_i$ whose union contains A_i. If we take $\mathcal{U}^*$ to be the union of all these subfamilies then it is finite and its union contains $\cup_i A_i$. Therefore the latter is compact.■

7. We need to show that if $F \subset X \times Y$ is closed then $\pi_X[F]$ is closed in X, and as usual it is enough to show that the complement is open. Suppose that $x \notin \pi_X[F]$. The latter implies that $\{x\} \times Y$ is contained in the open subset $X \times Y - F$, and by the Tube Lemma one can find an open set $V_x \subset X$ such that $x \in V$ and $V_x \times Y \subset X \times Y - F$. But this means that the open set $V_x \subset X$ lies in the complement of $\pi_X[F]$, and since one has a conclusion of this sort for each such x it follows that the complement is open as required.■

8. ($\Longrightarrow$) We shall show that the complement of the graph is open, and this does not use the compactness condition on Y (although it does use the Hausdorff property). Suppose we are given (x, y) such that $y \neq x$. Then there are disjoint open sets V and W in Y such that $y \in V$ and $f(x) \in W$. Since $f^{-1}[W]$ is open in X and contains x, there is an open set U containing x such that $f[U] \subset W$. It follows that $U \times V$ is an open subset of $X \times Y$ that is disjoint from Γ_f. Since we have such a subset for each point in the complement of Γ_f it follows that $X \times Y - \Gamma_f$ is open and that Γ_f is closed.■

($\Longleftarrow$) As in a previous exercises let $\gamma : X \to \Gamma_f$ be the graph map and let $j : \Gamma_f \to X \times Y$ be inclusion. General considerations imply that the map $\pi_X \circ j$ is continuous and 1–1 onto, and γ is the associated set-theoretic inverse. If we can prove that $\pi_X \circ j$ is a homeomorphism, then γ will also be a homeomorphism and then f will be continuous by a previous exercise. The map in question will be a homeomorphism if it is closed, and it suffices to check that it is a composite of closed mappings. By hypothesis j is the inclusion of a closed subset and therefore j is closed, and the preceding exercise shows that π_X is closed. Therefore the composite is a homeomorphism as claimed and the mapping f is continuous.■

Problem from Munkres, § 27, *pp.* 177 – 178

2. (a) The function of x described in the problem is continuous, so its set of zeros is a closed set. This closed set contains A so it also contains $\overline{A}$. On the other hand if $x \notin \overline{A}$ then there is an $\varepsilon > 0$ such that $N_\varepsilon(x) \subset X - \overline{A}$, and in this case it follows that $\mathbf{d}(x, A) \geq \varepsilon > 0$.■

(b) The function $f(a) = \mathbf{d}(x,a)$ is continuous and $\mathbf{d}(x,A)$ is the greatest lower bound for its set of values. Since A is compact, this greatest lower bound is a minimum value that is realized at some point of A.■

(c) The union is contained in $U(A,\varepsilon)$ because $\mathbf{d}(x,a) < \varepsilon$ implies $\mathbf{d}(x,A) < \varepsilon$. To prove the reverse inclusion suppose that y is a point such that $\delta = \mathbf{d}(y,A) < \varepsilon$. It then follows that there is some point $a \in A$ such that $\mathbf{d}(y,a) < \varepsilon$ because the greater than the greatest lower bound of all possible distances. The reverse inclusion is an immediate consequence of the existence of such a point a.■

(d) Let $F = X - U$ and consider the function $g(a) = \mathbf{d}(a,F)$ for $a \in A$. This is a continuous function and it is always positive because $A \cap F = \emptyset$. Therefore it takes a positive minimum value, say ε. If $y \in A \subset U(A,\varepsilon)$ then $\mathbf{d}(a,y) < \varepsilon \leq \mathbf{d}(a,F)$ implies that $y \notin F$, and therefore $U(A,\varepsilon)$ is contained in the complement of F, which is U.■

(e) Take X to be all real numbers with positive first coordinate, let A be the points of X satisfying $y = 0$, and let U be the set of all points such that $y < 1/|x|$. Then for every $\varepsilon > 0$ there is a point not in U whose distance from A is less than ε. For example, consider the points $(2n, 1/n)$.■

Additional exercises

1. Each of the subsets X_n is compact by an inductive argument, and since X is Hausdorff each one is also closed. Since each set in the sequence contains the next one, the intersection of finitely many sets $X_{k(1)}, \cdots, X_{k(n)}$ in the collection is the set $X_{k(m)}$ where $k(m)$ is the maximum of the $k(i)$. Since X is compact, the Finite Intersection Property implies that the intersection A of these sets is nonempty. We need to prove that $f(A) = A$. By construction A is the set of all points that lie in the image of the k-fold composite $\circ^k f$ of f with itself. To see that f maps this set into itself note that if $a = [\circ^k f](x_k)$ for each positive integer k then $f(a) = [\circ^k f](\, f(x_k)\,)$ for each k. To see that f maps this set onto itself, note that $a = [\circ^k f](x_k)$ for each positive integer k implies that

$$a = f\left([\circ^k f](x_{k+1})\right)$$

for each k.■

2. (a) In this situation it is convenient to work with topologies in terms of their closed subsets. Let $\mathcal{F}$ be the family of closed subsets of X associated to $\mathbf{T}$ and let $\mathcal{F}^*$ be the set of all subsets E such that $E \cap C$ is closed in X for every compact subset $C \subset X$. If E belongs to $\mathcal{F}$ then $E \cap C$ is always closed in X because C is closed, so $\mathcal{F} \subset \mathcal{F}^*$. We claim that $\mathcal{F}^*$ defines a topology on X and X is a Hausdorff k-space with respect to this topology.

The empty set and X belong to $\mathcal{F}^*$ because they already belong to $\mathcal{F}$. Suppose that E_α belongs to $\mathcal{F}^*$ for all α; we claim that for each compact subset $C \subset X$ the set $C \cap \bigcap_\alpha E_\alpha$ is $\mathcal{F}$-closed in X. This follows because

$$C \;\cap\; \bigcap_\alpha E_\alpha \;\;=\;\; \bigcap_\alpha (C \cap E_\alpha)$$

and all the factors on the right hand side are $\mathcal{F}$-closed (note that they are compact). To conclude the verification that $\mathcal{F}^*$ is a topology, suppose that E_1 and E_2 belong to $\mathcal{F}^*$. Once again let $C \subset X$ be compact, and observe that the set-theoretic equation

$$C \cap (E_1 \cup E_2) \;\;=\;\; (C \cap E_1) \;\cup\; (C \cap E_2)$$

implies the right hand side is $\mathcal{F}$-closed if E_1 and E_2 are.

Therefore $\mathcal{F}^*$ defines the closed subspaces of a topological space; let $\mathbf{T}^\kappa$ be the associated family of open sets. It follows immediately that the latter contains $\mathbf{T}$, and one obtains a Hausdorff space by the following elementary observation: *If* $(X, \mathbf{T})$ *is a Hausdorff topological space and* $\mathbf{T}^*$ *is a topology for* X *containing* $\mathbf{T}$*, then* $(X, \mathbf{T}^*)$ *is also Hausdorff.* This is true because the disjoint open sets in $\mathbf{T}$ containing a pair of disjoint points are also (disjoint) open subsets with respect to $\mathbf{T}^*$ containing the same respective points.

We now need to show that $\mathcal{K}(X) = (X, \mathbf{T}^\kappa)$ is a k-space and the topology is the unique minimal one that contains $\mathbf{T}$ and has this property. Once again we switch over to using the closed subsets in all the relevant topologies. The most crucial point is that a subset $D \subset X$ is $\mathcal{F}$-compact if and only if it is $\mathcal{F}^*$-compact; by construction the identity map from $[X, \mathcal{F}^*]$ to $[X, \mathcal{F}]$ is continuous (brackets are used to indicate the subset families are the closed sets), so if D is compact with respect to $\mathcal{F}^*$ it image, which is simply itself, must be compact with respect to $\mathcal{F}$. How do we use this? Suppose we are given a subset $B \subset X$ such that $B \cap D$ is $\mathcal{F}^*$-closed for every $\mathcal{F}^*$-compact subset D. Since the latter is also $\mathcal{F}$-compact and the intersection $B \cap D$ is $\mathcal{F}^*$ compact (it is a closed subspace of a compact space), we also know that $B \cap D$ is $\mathcal{F}$-compact and hence $\mathcal{F}$-closed. Therefore it follows that $\mathcal{F}^*$ is a k-space topology. If we are given the closed subsets for any Hausdorff k-space topology $\mathcal{E}$ containing $\mathcal{F}$, then this topology must contain all the closed sets of $\mathcal{F}^*$. Therefore the latter gives the unique minimal k-space topology containing the topology associated to $\mathcal{F}$.■

(b) Suppose that $F \subset Y$ has the property that $F \cap C$ is closed in Y for all compact sets $C \subset Y$. We need to show that $f^{-1}[F] \cap D$ is closed in X for all compact sets $D \subset X$.

If F and D are as above, then $f[D]$ is compact and by the assumption on f we know that

$$f^{-1}\,[\,F \cap f[D]\,] = f^{-1}[F] \cap f^{-1}\,[\,f[D]\,)$$

is closed in X with respect to the original topology. Since $f^{-1}\,[\,f(D]\,)$ contains the closed compact set D we have

$$f^{-1}[F] \cap f^{-1}\,(f[D]) \cap D \quad = \quad f^{-1}[F] \cap D$$

and since the left hand side is closed in X the same is true of the right hand side. But this is what we needed to prove.■

3. (a) ($\Longrightarrow$) The sequence of open subsets has a maximal element; let U_N be this element. Then $n \geq N$ implies $U_N \subset U_n$ by the defining condition on the sequence, but maximality implies the reverse inclusion. Thus $U_N = U_n$ for $n \geq N$.■

($\Longleftarrow$) Suppose that the Chain Condition holds but there is a nonempty family $\mathcal{U}$ of open subsets with no maximal element. If we pick any open set U_1 in this family then there is another open set U_2 in the family that properly contains U_2. Similarly, there is another open subset U_3 in the family that properly contains U_2, and we can inductively construct an ascending chain of open subspaces such that each properly contains the preceding ones. This contradicts the Ascending Chain Condition. Therefore our assumption that $\mathcal{U}$ had no maximal element was incorrect.■

(b) ($\Longrightarrow$) Take an open covering $\{\,U_\alpha\,\}$ of U for which each open subset in the family is nonempty, and let $\mathcal{W}$ be the set of all finite unions of subsets in the open covering. By definition this family has a maximal element, say W. If $W = U$ then U is compact, so suppose W is properly contained in U. Then if $u \in U - W$ and U_0 is an open set from the open covering that contains u, it will follow that the union $W \cap U_0$ is also a finite union of subsets from the open covering and it properly contains the maximal such set W. This is a contradiction, and it arises from our assumption that W was properly contained in U. Therefore U is compact.■

($\Longleftarrow$) We shall show that the Ascending Chain Condition holds. Suppose that we are given an ascending chain

$$U_1 \;\subset\; U_2 \;\subset\; \cdots$$

and let $W = \cup_n U_n$. By our hypothesis this open set is compact so the open covering $\{ U_n \}$ has a finite subcovering consisting of $U_{k(i)}$ for $1 \le i \le m$. If we take N to be the maximum of the $k(i)$'s it follows that $W = U_N$ and $U_n = U_N$ for $n \ge N$.■

(c) We begin by verifying the statement in the hint. If U is open in a noetherian Hausdorff space X, then U is compact and hence U is also closed (since X is Hausdorff). Since U is Hausdorff, one point subsets are closed and their complements are open, so the complements of one point sets are also closed and the one point subsets are also open. Thus a noetherian Hausdorff space is discrete. On the other hand, an infinite discrete space does not satisfy the Ascending Chain Condition (pick an infinite sequence of distinct points x_k and let U_n be the first n points of the sequence. Therefore a noetherian Hausdorff space must also be finite.■

(d) Suppose $Y \subset X$ where X is noetherian. Let $\mathcal{V} = \{ V_\alpha \}$ be a nonempty family of open subspaces of Y, write $V_\alpha = U_\alpha \cap Y$ where U_α is open in X, and let $\mathcal{U} = \{ U_\alpha \}$. Since X is noetherian, this family has a maximal element U^*, and the intersection $V^* = U^* \cap Y$ will be a maximal element of $\mathcal{V}$.■

4. Since X is Hausdorff every one point set is closed, and this implies that $\mathbf{L}(A)$ is closed in X. We are assuming that $\overline{A}$ is compact, and since the closed subset $\mathbf{L}(A)$ is contained in $\overline{A}$ it follows that $\mathbf{L}(A)$ is also compact.■

5. (i) Since X is compact we know that $f[X]$ is compact in $\mathbb{R}$ with respect to the lower semicontinuity topology. Every subset $C \subset \mathbb{R}$ has an open covering in this topology consisting of proper open subsets $(b_\alpha, \infty) \cap C$ because every point lies in a proper open subset (with respect to the lower semicontinuity topology). If C is compact, this means that C is contained in a finite union of these sets:

$$C \quad \subset \quad \bigcup_{j=1}^{k} (b_j, \infty)$$

The right hand side is equal to (b^*, ∞) where b^* is the smallest in the finite collection of numbers $\{b_j\}$, and this implies that $x > b^*$ for all $x \in C$. In particular, this applies to $f[X]$ if f is lower semicontinuous and X is compact, and hence we have shown that $f[X]$ has a lower bound.■

(ii) To continue the discussion from (i), let m be the greatest lower bound of $f[X]$. Then for each $n > 0$ the set $F_n = f^{-1}\left[(-\infty, m + \frac{1}{n}]\right]$ is nonempty, for there will be some $x \in X$ such that $f(x) < m - \frac{1}{n}$; since $(-\infty, m + \frac{1}{n}]$ is closed in the lower semicontinuity topology, it follows that F_n is also closed. Since $F_n \supset F_{n+1}$ for all n, this yields a nested sequence of closed subspaces $F_1 \supset \cdots F_n \supset F_{n+1} \cdots$ such that each F_n is nonempty. By the compactness of X we know that their intersection must also be nonempty.

Let $x_0 \in \cap_n F_n$. Then since $x_0 \in F_k$ for each k we have $f(x) \le m + \frac{1}{k}$ for each k. This implies that $f(x) \le m$. However, we also know that m is a lower bound for $f[X]$, and therefore we must have $f(x_0) = m$; in other words, f takes a minimum value at x_0.■

III.2 : Complete metric spaces

Problems from Munkres, § 43, pp. 270 – 271

(a) Let $\{ x_n \}$ be a Cauchy sequence in X and choose M so large that $m, n \ge M$ implies $\mathbf{d}(x_m.x_n) < \varepsilon$. Then all of the terms of the Cauchy sequence except perhaps the first $M - 1$ lie in the closure of $N_\varepsilon(x_M)$, which is compact. Therefore it follows that the sequence has a convergent subsequence $\{ x_{n(k)} \}$. Let y be the limit of this subsequence; we need to show that y is the limit of the entire sequence.

Let $\eta > 0$ be arbitrary, and choose $N_1 \geq M$ such that $m, n \geq N_1$ implies $\mathbf{d}(x_m, x_n) < \eta/2$. Similarly, let $N_2 \geq M$ be such that $n(k) > N_2$ implies $\mathbf{d}(x_{n(k)}, y) < \eta/2$. If we take N to be the larger of N_1 and N_2, and application of the Triangle Inequality shows that $n \leq N$ implies $\mathbf{d}(x_n, y) < \eta$. Therefore y is the limit of the given Cauchy sequence and X is complete.■

(*b*) Take $U \subset \mathbb{R}^2$ to be the set of all points such that $xy < 1$. This is the region "inside" the hyperbolas $y = \pm 1/x$ that contains the origin. It is not closed in $\mathbb{R}^2$ and therefore cannot be complete. However, it is open and just like all open subsets U of $\mathbb{R}^2$ if $x \in X$ and $N_\varepsilon(x) \subset U$ then $N_{\varepsilon/2}(x)$ has compact closure in U.■

3. (*b*) By the symmetry of the problem it is enough to show that if $(X, \mathbf{d})$ is complete then so is $(X, \mathbf{e})$. Suppose that $\{ x_n \}$ is a Cauchy seuence with respect to $\mathbf{e}$; we claim it is also a Cauchy sequence with respect to $\mathbf{d}$. Let $\varepsilon > 0$, and take $\delta > 0$ such that $\mathbf{e}(u, v) < \delta$ implies $\mathbf{d}(u, v) < \varepsilon$. If we choose M so that $m, n \geq M$ implies $\mathbf{e}(x_n, x_m) < \delta$, then we also have $\mathbf{d}(x_n, x_m) < \varepsilon$. Therefore the original Cauchy sequence with respect to $\mathbf{e}$ is also a Cauchy sequence with respect to $\mathbf{d}$. By completeness this sequence has a limit, say y, with respect to $\mathbf{d}$, and by continuity this point is also the limit of the sequence with respect to $\mathbf{e}$.■

6. (*a*) This follows because a closed subspace of a complete metric space is complete.■

(*c*) The image of the function f is the graph of ϕ, and general considerations involving graphs of continuous functions show that f maps U homeomorphically onto its image. If this image is closed in $X \times \mathbb{R}$ then by **(a)** we know that U is topologically complete, so we concentrate on proving that $f(U) \subset X \times \mathbb{R}$ is closed. The latter in turn reduces to proving that the image is closed in the complete subspace $\overline{U} \times \mathbb{R}$, and as usual one can prove this by showing that the complement of $f[U]$ is open. The latter in turn reduces to showing that if $(x, t) \in \overline{U} \times \mathbb{R} - f[U]$ then there is an open subset containing (x, t) that is disjoint from U.

Since ϕ is continuous it follows immediately that the graph of ϕ is closed in the open subset $U \times \mathbb{R}$. Thus the open set $U \times \mathbb{R} - f[U]$ is open in $\overline{U} \times \mathbb{R}$, and it only remains to consider points in $\overline{U} - U \times \mathbb{R}$. Let (x, t) be such a point. In this case one has $\mathbf{d}(x, X - U) = 0$. There are three cases depending on whether t is less than, equal to or greater than 0. In the first case we have that $(x, t) \in \overline{U} \times (-\infty, 0)$ which is open in $\overline{U} \times \mathbb{R}$ and contains no points of $f[U]$ because the second coordinates of points in the latter set are always positive. Suppose now that $t = 0$. Then by continuity of distance functions there is an open set $V \subset \overline{U}$ such that $x \in V$ and $\mathbf{d}(y, X - U) < 1$ for all $y \in V$. It follows that $V \times (-1, 1)$ contains (x, t) and is disjoint from $f[U]$. Finally, suppose that $t > 0$. Then by continuity there is an open set $V \subset \overline{U}$ such that $x \in V$ and

$$\mathbf{d}(y, X - U) \quad < \quad \frac{2}{3t}$$

for all $y \in V$. It follows that $V \times (t/2, 3t/2)$ contains (x, t) and is disjoint from $f[U]$, and this completes the proof of the final case.■

Additional exercises

1. Follow the hint, and try to see what a function in the intersection would look like. In the first place it has to satisfy $f(0) = 1$, but for each $n > 0$ it must be zero for $t \geq 1/n$. The latter means that the $f(t) = 0$ for all $t > 0$. Thus we have determined the values of f everywhere, but the function we obtained is not continuous at zero. Therefore the intersection is empty. Since every function in the set A_n takes values in the closed unit interval, it follows that if f and g belong to A_n then $\|f - g\| \leq 1$ and thus the diameter of A_n is at most 1 for all n. In fact, the diameter is exactly 1 because $f(0) = 1$.

For the sake of completeness, we should note that each set A_n is nonempty. One can construct a "piecewise linear" function in the set that is zero for $t \geq 1/n$ and decreases linearly from the 1 to 0 as t increases from 0 to $1/n$. (Try to draw a picture of the graph of this function!)■

2. For each positive integer k let $H_k(y)$ be the vector whose first k coordinates are the same as those of y and whose remaining coordinates are zero, and let $T_k = I - H_k$ (informally, these are "head" and "tail" functions). Then H_k and T_k are linear transformations and $|H_k(y)|,\ |T_k(y)|\ \leq |y|$ for all y.

Since Cauchy sequences are bounded there is some $B > 0$ such that $|x_n| \leq B$ for all n.

Let x be given as in the hint. We claim that $x \in \ell^2$; by construction we know that $H_k(x) \in \ell^2$ for all k. By the completeness of $\mathbb{R}^M$ we know that $\lim_{n\to\infty} H_k(x_n) = H_k(x)$ and hence there is an integer P such that $n \geq P$ implies $|H_k(x) - H_k(x_n)| < \varepsilon$. If $n \geq P$ we then have that

$$|H_k(x)| \ \leq \ |H_k(x_n)| + |H_k(x) - H_M(k_n)| \ \leq \ |x_n| + |H_M(x) - H_M(x_n)| \ < \ B + \varepsilon \ .$$

By construction $|x|$ is the least upper bound of the numbers $|H_k(x)|$ if the latter are bounded, and we have just shown the latter are bounded. Therefore $x \in \ell^2$; in fact, the argument can be pushed further to show that $|x| \leq B$, but we shall not need this.

We must now show that x is the limit of the Cauchy sequence. Let $\varepsilon > 0$ and choose M this time so that $n,\, m \geq M$ implies $|x_m - x_n| < \varepsilon/6$. Now choose N so that $k \geq N$ implies $|T_k(x_M)| < \varepsilon/6$ and $|T_k(x)| < \varepsilon/3$; this can be done because the sums of the squares of the coordinates for x_M and x are convergent. If $n \geq M$ then it follows that

$$|T_k(x_n)| \ \leq \ |T_k(x_M)| + |T_k(x_n) - T_k(x_M)| = \ |T_k(x_n)| \ \leq \ |T_k(x_M)| + |T_k(x_n - x_M)| \ \leq$$

$$|T_k(x_M)| + |x_n - x_M| \ \leq \ \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \ = \ \frac{\varepsilon}{3} \ .$$

Choose P so that $P \geq M + N$ and $n \geq P$ implies $|H_N(x) - H_N(x_n)| < \frac{\varepsilon}{3}$. If $n \geq P$ we then have

$$|x - x_n| \ \leq \ |H_N(x) - H_N(x_n)| + |T_N(x) - T_N(x_n)| \ \leq$$

$$|H_N(x) - H_N(x_n)| \ + \ |T_N(x)| \ + \ |T_N(x_n)| \ < \ \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \ = \ \varepsilon \ .$$

This completes the proof.■

III.3 : Implications of completeness

Problems from Munkres, § 48, pp. 298 – 300

1. If the interior of the closure of B_n is empty, then B_n is nowhere dense. Thus if the interior of the closure of each B_n is empty then X is of the first category.■

2. View the real numbers as a vector space over the rationals. The previous argument on the existence of bases implies that the set $\{1\}$ is contained in a basis (work out the details!). Let B be such a basis (over the rationals), and let W be the rational subspace spanned by all the remaining vectors in the basis. Then $\mathbb{R}$ is the union of the cosets $c + W$ where c runs over all rational numbers, and this is a countable union. We claim that each of these cosets has a nonempty interior. To see this, note that each coset contains exactly one rational number, while if there interior were nonempty it would contain an open interval and every open interval contains infinitely many rational numbers (*e.g.*, if $c \in (a,b)$ there is a strictly decreasing sequence of rational numbers $r_n \in (c,b)$ whose limit is c).■

4. According to the hint, we need to show that if $\{ V_n \}$ is a sequence of open dense subsets in X, then $(\cap_n V_n)$ is dense.

Let $x \in X$, and let W be an open neighborhood of x that is a Baire space. We claim that the open sets $V_n \cap W$ are all dense in W. Suppose that W_0 is a nonempty open subset of W; then W_0 is also open in X, and since $\cap_n V_n$ is dense in X it follows that

$$W_0 \cap (W \cap V_n) = (W_0 \cap W) \cap V_n = W_0 \cap V_n \neq \emptyset$$

for all n. Therefore $V_n \cap W$ is dense in W. Since W is a Baire space and, it follows that

$$\bigcap_n (V_n \cap W) = W \cap \left(\bigcap_n V_n \right)$$

is dense in W.

To show that $(\cap_n V_n)$ is dense in X, let U be a nonempty open subset of X, let $a \in U$, let W_a be an open neighborhood of a that is a Baire space, and let $U_0 = U \cap W$ (hence $a \in U_0$). By the previous paragraph the intersection

$$U_0 \cap \left(\bigcap_n (W \cap V_n) \right)$$

is nonempty, and since this intersection is contained in

$$U \quad \cap \quad \left(\bigcap_n V_n \right)$$

it follows that the latter is also nonempty, which implies that the original intersection $(\cap_n V_n)$ is dense.■

Problem from Munkres, § 27, pp. 178 − 179

6. PRELIMINARY OBSERVATION. Each of the intervals that is removed from A_{n-1} to construct A_n is entirely contained in the former. One way of doing this is to partition $[0,1]$ into the 3^n intervals

$$\left[\frac{b}{3^n}, \frac{b+1}{3^n} \right]$$

where b is an integer between 0 and $3^n - 1$. If we write down the unique 3-adic expansion of b as a sum

$$\sum_{i=0}^{n-1} a_i \, 3^i$$

where $b_i \in \{0,1,2\}$, then A_k consists of the intervals associated to numbers b such that none of the coefficients b_i is equal to 1. Note that these intervals are pairwise disjoint; if the interval corresponding to b lies in A_k then either the interval corresponding to $b+1$ or $b-1$ is not one of the intervals that are used to construct A_k (the base 3 expansion must end with a 0 or a 2, and thus one of the adjacent numbers has a base 3 expansion ending in a 1). The inductive construction of A_n reflects the fact that for each k the middle third is removed from the closed interval

$$\left[\frac{k}{3^{n-1}}, \frac{k+1}{3^{n-1}}\right] .$$

(a) By induction A_k is a union of 2^k pairwise disjoint closed intervals, each of which has length 3^{-k}; at each step one removes the middle third from each of the intervals at the previous step. Suppose that $K \subset C$ is nonempty and connected. Then for each n the set K must lie in one of the 2^n disjoint intervals of length 3^{-n} in A_n. Hence the diameter of K is $\leq 3^{-n}$ for all $n \geq 0$, and this means that the diameter is zero; *i.e.*, K consists of a single point.■

(b) By construction C is an intersection of closed subsets o the compact space $[0, 1]$, and therefore it is closed and hence compact.■

(c) If we know that the left and right endpoints for the intervals comprising A_n are also (respectively) left and right hand endpoints for intervals comprising A_{n+1}, then by induction it will follow that they are similar endpoints for intervals comprising A_{n+k} for all $k \geq 0$ and therefore they will all be points of C. By the descriptions given before, the left hand endpoints for A_n are all numbers of the form $b/3^n$ where b is a nonnegative integer of the form

$$\sum_{i=0}^{n-1} b_i \, 3^i$$

with $b_i = 0$ or 2 for each i, and the right hand endpoints have the form $(b+1)/3^n$ where b has the same form. If b has the indicated form then

$$\frac{b}{3^n} \quad = \quad \frac{3b}{3^{n+1}} \ , \quad \text{where} \quad 3b = \quad \sum_{i=1}^{n} b_{i-1} \, 3^i$$

shows that $b/3^n$ is also a left hand endpoint for one of the intervals comprising A_{n+1}. Similarly, if $(b+1)/3^n$ is a right hand endpoint for an interval in A_n and b is expanded as before, then the equations

$$\frac{b+1}{3^n} \quad = \quad \frac{3b+3}{3^{n+1}}$$

and

$$(3b+3) - 1 \quad = \quad 3b+2 \quad = \quad 2 + \sum_{i=1}^{n} b_{i-1} \, 3^i$$

show that $(b+1)/3^n$ is also a right hand endpoint for one of the intervals comprising A_{n+1}.■

(d) If $x \in C$ then for each n one has a unique closed interval J_n of length 2^{-n} in A_n such that $x \in J_n$. Let $\lambda_n(x)$ denote the left hand endpoint of that interval unless x is that point, and let $\lambda_n(x)$ be the right hand endpoint in that case; we then have $\lambda_n(x) \neq x$. By construction $\left| \lambda_n(x) - x \right| < 2^{-n}$ for all n, and therefore $x = \lim_{n \to \infty} \lambda_n(x)$. On the other hand, by the preceding portion of this problem we know that $\lambda_n(x) \in C$, and therefore we have shown that x is a limit point of C, which means that x is not an isolated point.■

(e) One way to do this is by using (d) and the Baire Category Theorem. By construction C is a compact, hence complete, metric space. It is infinite by (c), and every point is a limit point by (d). Since a countable complete metric space has isolated points, it follows that C cannot be countable.

In fact, one can show that $|C| = 2^{\aleph_0}$. The first step is to note that *if* $\{\, a_k \,\}$ *is an infinite sequence such that* $a_k \in \{0, 2\}$ *for each* k*, then the series*

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

converges and its sum lies in C.

This assertion may be verified as follows: The infinite series converges by a comparison test with the convergent series such that $a_k = 2$ for all k. Given a point as above, the partial sum

$$\sum_{k=1}^{n} \frac{a_k}{3^k}$$

is a left hand endpoint for one of the intervals comprising A_n. The original point will lie in A^n if the sum of the rest of the terms is $\leq 1/3$. But

$$\sum_{k=n+1}^{\infty} \frac{a_k}{3^k} \;\leq\; \sum_{k=n}^{\infty} \frac{2}{3^k} \;=\; \frac{2}{3^{n+1}} \cdot \frac{1}{(1 - \frac{1}{3})} \;=\; \frac{1}{3}$$

so the point does lie in A_n. Since n was arbitrary, this means that the sum lies in $\cap_n \, A_n = C$.

Returning to the original problem of determining $|C|$, we note that the set $\mathcal{A}$ of all sequences described in the assertion is in a natural 1–1 correspondence with the set of all functions from the positive integers to $\{0, 1\}$. Let $\mathcal{A}_0$ be the set of all functions whose values are nonzero for infinitely many values of n, and let $\mathcal{A}_1$ be the functions that are equal to zero for all but finitely many values of n. We then have that $|\mathcal{A}_1| = \aleph_0$ and $\mathcal{A}_0$ is infinite (why?). The map sending a function in $\mathcal{A}_0$ to the associated sum of an infinite series is 1–1 (this is just a standard property of base N expansions — work out the details), and therefore we have

$$|\mathcal{A}_0| \;\leq\; |C| \;\leq\; |\mathbb{R}| \;=\; 2^{\aleph_0}$$

and

$$|\mathcal{A}_0| \;=\; |\mathcal{A}_0| + \aleph_0 \;=\; |\mathcal{A}_0| + |\mathcal{A}_1| \;=\; |\mathcal{A}| \;=\; 2^{\aleph_0}$$

which combine to imply $|C| = 2^{\aleph_0}$.■

Additional exercises

1. Suppose that the conclusion is false: *i.e.*, A is not nowhere dense in X and B is not nowhere dense in Y. Then there are nonempty open sets U and V contained in the closures of A and B respectively, and thus we have

$$\emptyset \;\neq\; U \times V \;\subset\; \overline{A} \times \overline{B} \;=\; \overline{A \times B}$$

and therefore $A \times B$ is not nowhere dense in $X \times Y$.

To see that we cannot replace "or" with "and" take $X = Y = \mathbb{R}$ and let A and B be equal to $[0, 1]$ and $\{0\}$ respectively. Then A is not nowhere dense in X but $A \times B$ is nowhere dense in $X \times Y$.■

2. The space $\mathbb{R}^\infty$ has this property because it is the union of the closed nowhere dense subspaces A_n that are defined by the condition $x_i = 0$ for $i > n$.■

3. %vfil

First of all, the map f is 1–1 onto; we are given that it is onto, and it is 1–1 because $u \neq v$ implies $\mathbf{d}(\, f(u),\, f(v)\,) > \mathbf{d}(u,v) > 0$. Therefore f has an inverse, at least set-theoretically, and we denote f^{-1} by T.

We claim that T satisfies the hypotheses of the Contraction Lemma. The proof of this begins with the relations

$$\mathbf{d}(\, T(u),\, T(v)\,) = \mathbf{d}(\, f^{-1}(u),\, f^{-1}(v)\,) = \frac{1}{C1}\,\mathbf{d}\left(\, f\left(f^{-1}(u)\right),\, f\left(f^{-1}(v)\right)\,\right) = \mathbf{d}(u,v)\ .$$

Since $C > 1$ it follows that $0 < 1/C < 1$ and consequently the hypotheses of the Contraction Lemma apply to our example.

Therefore T has a unique fixed point p; we claim it is also a fixed point for f. We shall follow the hint. Since T is 1–1 and onto, it follows that $x = T(\,T^{-1}(x)\,)$ and that $T(x) = x \implies x = T^{-1}(x)$; the converse is even easier to establish, for if $x = T^{-1}(x)$ the application of T yields $T(x) = x$. Since there is a unique fixed point p such that $T(p) = p$, it follows that there is a unique point, in fact the same one as before, such that $p = T^{-1}(p)$, which is equal to $f(p)$ by definition.■

THE CLASSICAL EUCLIDEAN CASE.

This has two parts. The first is that every expanding similarity of $\mathbb{R}^n$ is expressible as a so-called affine transformation $T(v) = cAv + b$ where A is given by an orthogonal matrix. The second part is to verify that each transformation of the type described has a unique fixed point. By the formula, the equation $T(x) = x$ is equivalent to the equation $x = cAx + b$, which in turn is equivalent to $(I - cA)x = b$. The assertion that T has a unique fixed point is equivalent to the assertion that this linear equation has a unique solution. The latter will happen if $I - cA$ is invertible, or equivalently if $\det(I - cA) \neq 0$, and this is equivalent to saying that c^{-1} is not an eigenvalue of A. But if A is orthogonal this means that $|Av| = |v|$ for all v and hence the only possible eigenvalues are ± 1; on the other hand, by construction we have $0 < c^{-1} < 1$ and therefore all of the desired conclusions follow. The same argument works if $0 < c < 1$, the only change being that one must substitute $c^{-1} > 1$ for $0 < c^{-1} < 1$ in the preceding sentence.■

4. We need to show that φ maps $[\sqrt{a}, x_0]$ into itself and that the absolute value of its derivative takes a maximum value that is less than 1.

In this example, the best starting point is the computation of the derivative, which is simply an exercise in first year calculus:

$$\varphi'(x) \quad = \quad \frac{1}{2}\left(1 - \frac{a}{x^2}\right)$$

This expression is an increasing function of x over the set $[\sqrt{a}, +\infty)$; its value at $\sqrt{a}$ is 0 and the limit at $+\infty$ is 1/2. In particular, the absolute value of the derivative on $[\sqrt{a}, x_0]$ is less than 1/2, and by the Mean Value Theorem the latter in turn implies that φ maps the interval in question to

$$\left[\sqrt{a}, \frac{\sqrt{a} + x_0}{2}\right]$$

which is contained in $[\sqrt{a}, x_0]$.■

5. (i) We shall first prove that the interior of H in $\mathbb{R}^n$ is empty. Suppose to the contrary that there is some $p \in H$ and some $\varepsilon > 0$ such that $N_\varepsilon(p) \subset H$. Then by the definition of H we know that $F = 0$ on $N_\varepsilon(p)$. However, we have

$$F(p + ta) \quad = \quad F(p) \ + \ t|a|^2 \quad = \quad t|a|^2$$

so $p + ta \notin H$ for $t \neq 0$; since $t < \varepsilon/|a|$ implies that $p + ta \in N_\varepsilon(p)$, we cannot have $N_\varepsilon(p) \subset H$. This contradiction implies that the interior of H is empty. Note also that H is closed because it is the zero set of a continuous real valued function.

We shall now prove that $\mathbb{R}^n - H$ is dense in $\mathbb{R}^n$. If this were not the case, then the complement of the closure contains some open subset of the form $N_r(q)$, and this open subset would have to be contained in H. Since H has an empty interior, this cannot happen, and therefore $\mathbb{R}^n - H$ must be dense in $\mathbb{R}^n$.■

(ii) We have $\mathbb{R}^n - \cup_i H_i = \cap_i \ (\mathbb{R}^n - H_i)$, and by (i) we know that each factor in the intersection is an open dense subset. Since a finite intersection of open and dense subsets is dense, it follows that the set in question is dense.■

6. (i) The function in question can be rewritten in the vector form $F(x) = \langle a, x\rangle - b$, where $a = (a_1, \ \cdots \ , a_n)$ and $\langle \ , \ \rangle$ denotes the usual dot product. Suppose that x_0 lies in H, and consider the curve $x_0 + ta$ for $t \in R$. Then $F(x + ta) = \langle a, x + ta\rangle + b = \langle a, x_0\rangle + t|a|^2 - b$, and since x_0 lies on the hyperplane defined by $F(x) = 0$ we have $F(x + ta) = t|a|^2$, where $a \neq 0$. This quantity is positive if $t > 0$ and negative if $t < 0$, so both H_+ and H_- are nonempty. These sets are open by continuity of F, so the only thing left to prove is that these subsets are convex. Following the hint, we shall first prove that $F(ty + (1-t)x) = t\,F(y) + (1-t)\,F(x)$ for all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$:

$$F(ty \ + \ (1-t)x) \ = \ \langle a, ty + (1-t)x\rangle \ - \ b \ = \ t\langle a, y\rangle \ + \ (1-t)\,\langle a, x\rangle \ - \ b \ =$$

$$t\langle a, y\rangle \ + \ (1-t)\,\langle a, x\rangle \ - \ \big(t\,b + (1-t)\,b\big) \ = \ t\,(\langle a, y\rangle \ - \ b) \ + \ (1-t)\,(\langle a, x\rangle \ - \ b) \ =$$

$$t\,F(y) \ + \ (1-t)\,F(x)$$

If $0 < t < 1$ and the values $F(x), F(y)$ are both positive or both negative, the identity and convexity of $(0, \infty) \subset \mathbb{R}$ imply that $F(ty + (1-t)x)$ is also positive or negative. Therefore both H_+ and H_- are convex.■

(ii) Suppose that $F(x) > 0 > F(y)$. We want to find some t such that $0 < t < 1$ and $0 = F(tx + (1-t)y) = t\,F(x) + (1-t)\,F(y)$. This equation has the solution

$$t \ = \ \frac{-F(y)}{F(x) - F(y)}$$

so we need to show that the right hand side lies between 0 and 1. Since $F(y) < 0$, the numerator is positive, and likewise for the denominator because $F(x) > 0$ and hence $F(x) - F(y) > F(x) > 0$. Furthermore, the latter chain of inequalities implies that $t < 1$.■

(iii) Let $\mathbf{H}$ denote the topology on $\mathbb{R}^n$ generated by half-spaces, and let $\mathbf{M}$ be the usual metric topology. Since half-spaces are open in the metric topology, it follows that $\mathbf{H} \subset \mathbf{M}$. To prove the reverse implication, note that it suffices to prove that the base for $\mathbf{M}$ consisting of the $\mathbf{d}^{\langle\infty\rangle}$ neighborhoods

$$\prod_{j=1}^{n} \ (a_i - \varepsilon, a_i + \varepsilon)$$

is contained in $\mathbf{H}$. This will follow if we can show that each of the displayed sets is a finite intersection of half-spaces, and the latter in turn follows because the subspace in the display is the intersection of the half-spaces defined by the strict inequalities $x_j - a_j > -\varepsilon$ and $x_j - a_j < \varepsilon$ for $1 \leq j \leq n$.■

Note. Many systems of synthetic axioms for classical geometries include some form of *Space Separation Axiom* which states that the complement of a hyperplane (a line in two dimensions, a plane in three dimensions, *etc.*) satisfies the conclusions of this exercise, and for such a system there is a natural synthetic topology generated by the half-spaces. One implication of the preceding exercise is that this synthetic topology coincides with the usual one if we are working with axioms for Euclidean geometry or closely related systems.

III.4 : Connected spaces

Problems from Munkres, § 23, p. 152

2. Let C be the connected component of $\cup_n A_n$ containing A_1. We shall prove by induction that $A_k \subset C$ for each positive integer k. It will follow that $C = \cup_n A_n$ and that the latter is connected.

Suppose that A_k is contained in C; we want to show that $A_{k+1} \subset C$. We are given that there is at least one point $p \in A_k \cup A_{k+1}$, and therefore we know that this union of the connected subsets A_k and A_{k+1} is connected. Since C is a connected component of $\cup_n A_n$ containing A_k, it follows that C must also contain the connected subset $A_k \cup A_{k+1}$, and hence it follows that $A_{k+1} \subset C$, completing the proof of the inductive step.■

3. Suppose that C is a nonempty open and closed subset of $Y = A \cup B$. Then by connectedness either $A \subset C$ or $A \cap C = \emptyset$; without loss of generality we may assume the first holds, for the argument in the second case will follow by interchanging the roles of C and $Y - C$. We need to prove that $C = Y$.

Since each A_α for each α we either have $A_\alpha \subset C$ or $A_\alpha \cap C = \emptyset$. In each case the latter cannot hold because $A \cap A_\alpha$ is a nonempty subset of C, and therefore A_α is contained in C for all α. Therefore $C = Y$ and hence Y is connected.■

4. Suppose that we can write $X = C \cup D$ where C and D are disjoint nonempty open and closed subsets. Since C is open it follows that either D is finite or all of X; the latter cannot happen because $X - D = C$ is nonempty. On the other hand, since C is closed it follows that either C is finite or $C = X$. Once again, the latter cannot happen because $X - C = D$ is nonempty. Thus X is a union of two finite sets and must be finite, which contradicts our assumption that X is infinite. This forces X to be connected.■

5. Suppose that C is a maximal connected subset; then C also has the discrete topology, and the only discrete spaces that are connected are those with at most one point.

There are many examples of totally disconnected spaces that are not discrete. The set of rational numbers and all of its subspaces are fundamental examples; here is the proof: Let x and y be distinct rational numbers with $x < y$. Then there is an irrational number r between them, and the identity

$$\mathbb{Q} = \Big(\mathbb{Q} \cap (-\infty, r)\Big) \quad \cup \quad \Big(\mathbb{Q} \cap (r, +\infty, r)\Big)$$

gives a separation of $\mathbb{Q}$, thus showing that x and y lie in different connected components of $\mathbb{Q}$. But x and y were arbitrary so this means that no pair of distinct points can lie in the same connected component of Q. The argument for subsets of $\mathbb{Q}$ proceeds similarly.■

9. A good way to approach this problem is to begin by drawing a picture in which $X \times Y$ is a square and $A \times B$ is a smaller concentric square. It might be helpful to work with this picture while reading the argument given here.

Let $x_0 \in X - A$ and $y_0 \in Y - B$, and let C be the connected component of (x_0, y_0) in $X \times Y - A \times B$. We need to show that C is the entire space, and in order to do this it is enough to show that given any other point (x, y) in the space there is a connected subset of $X \times Y - A \times B$ containing it and (x_0, y_0). There are three cases depending upon whether or not $x \in A$ or $y \in B$ (there are three options rather than four because we know that both cannot be true).

If $x \notin A$ and $y \notin B$ then the sets $X \times \{y_0\}$ and $\{x\} \times Y$ are connected subsets such that (x_0, y_0) and (x, y_0) lie in the first subset while (x, y_0) and (x, y) lie in the second. Therefore there is a connected subset containing (x, y) and (x_0, y_0) by Exercise 3.— Now suppose that $x \in A$ but $y \notin B$. Then the two points in question are both contained in the connected subset $X \times \{y\} \cup times\{x_0\} \times Y$. Finally, if $x \notin A$ but $y \in B$, then the two points in question are both contained in the connected subset $X \times \{y_0\} \cup \{x\} \times Y$. Therefore the set $X \times Y - A \times B$ is connected.■

12. [*Assuming Y is a closed subset of X. Munkres does not explicitly assume $Y \neq \emptyset$, but without this assumption the conclusion is false.*] Since $X - Y$ is open in X and A and B are disjoint open subsets of $X - Y$, it follows that A and B are open in X. The latter in turn implies that $X \cup A$ and $Y \cup B$ are both closed in X.

We shall only give the argument for $X \cup A$; the proof for $X \cup B$ is the similar, the only change being that the roles of A and B are interchanged. Once again, it might be helpful to draw a picture.

Suppose that C is a nonempty proper subset of $Y \cup A$ that is both open and closed, and let $D = (Y \cup A) - C$. One of the subsets C, D must contain some point of Y, and without loss of generality we may assume it is C. Since Y is connected it follows that all of Y must be contained in C. Suppose that $D \neq \emptyset$. Since D is closed in $Y \cup A$ and the latter is closed in X, it follows that D is closed in X. On the other hand, since D is open in $Y \cup A$ and disjoint from Y it follows that D is open in A, which is the complement of Y in $Y \cup A$. But A is open in X and therefore D is open in X. By connectedness we must have $D = X$, contradicting our previous observation that $D \cap Y = \emptyset$. This forces the conclusion that D must be empty and hence $Y \cup A$ is connected.■

Problems from Munkres, § 24, pp. 157 – 159

1. (*b*) Take $X = (0, 1) \cup (2, 3)$ and $Y = (0, 3)$, let f be inclusion, and let g be multiplication by $1/3$. There are many other examples. In the spirit of part **(a)** of this exercise, one can also take $X = (0, 1)$, $Y = (0, 1]$, f = inclusion and g = multiplication by $1/2$, and similarly one can take $X = (0, 1]$, $Y = [0, 1]$, f = inclusion and $g(t) = (t + 1)/2$.■

FOOTNOTE. Notwithstanding the sort of examples described in the exercises, a result of S. Banach provides a "resolution" of X and Y if there are continuous embeddings $f : X \to Y$ and $g : Y \to X$; namely, there are decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $X_1 \cap X_2 = Y_1 \cap Y_1 = \emptyset$ and there are homeomorphisms $X_1 \to X_2$ and $Y_1 \to Y_2$. The reference is S. Banach, *Un théorème sur les transformations biunivoques*, Fundamenta Mathematicæ **6** (1924), 236–239.

Additional exercises

1. Define a binary relation $\sim$ on X such that $u \sim v$ if and only if there are open subsets U and V in $\mathcal{U}$ such that $u \in U$, $v \in V$, and there is a sequence of open sets $\{U_0, U_1, \cdots U_n\}$ in $\mathcal{U}$ such that $U = U_0$, $V = U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for all i. This is an equivalence relation (verify this in detail!). Since every point lies in an open subset that belongs to $\mathcal{U}$ it follows that the equivalence classes are open. Therefore the union of all but one equivalence class is also open, and hence a single equivalence class is also closed in the space. If X is connected, this can only happen if there is exactly one equivalence class.■

2. This is similar to some arguments in the course notes and to the preceding exercise. The hypothesis that the relation is locally constant implies that equivalence classes are open. Therefore the union of all but one equivalence class is also open, and hence a single equivalence class is also closed in the space. If X is connected, this can only happen if there is exactly one equivalence class.■

3. The important point is that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$; given a point $x \in \mathbb{R}^n$ with coordinates x_i and $\varepsilon > 0$ we can choose rational numbers q_i such that $|q_i - x_i| < \varepsilon/n$; if y has coordinates q_i then $|x - y| < \varepsilon$ follows immediately. Since $\mathbb{R}^n$ is locally connected, the components of open sets are open, and therefore we conclude that every component of a nonempty open set $U \subset \mathbb{R}^n$ contains some point of $\mathbb{Q}^n$. Picking one such point for each component we obtain a 1–1 map from the set of components into the countable set $\mathbb{Q}^n$.

The Cantor set is an example of a closed subset of $\mathbb{R}$ for which the components are the one point subsets and there are uncountably many points.■

4. (*i*) FALSE. One simple counterexample is given by taking $X = Y = \mathbb{R}$ and $A = B = \{0\}$. Then $X \times Y - A \times B$ is merely $\mathbb{R}^2 - \{\mathbf{0}\}$, which is the image of the connected set $(0, \infty) \times [0, 2\pi]$ under the polar coordinate map sending (r, θ) to $(r \cos\theta, r \sin\theta)$.■

(*ii*) FALSE. Take $A = (0, 1]$ and $B = \{0, 1\}$ so that $A \cap B = \{1\}$ and $A \cup B = [0, 1]$.

(*iii*) TRUE. Given that the hypotheses are nearly the same as in (*ii*) but slightly stronger, one might guess this answer, but of course a proof is still needed.

We shall only prove that A is connected; the connectedness of B will then follow by interchanging the roles of A and B in the argument given here. — Suppose that A is not connected, and write it as a union of two nonempty closed subsets $A_1 \cup A_2$. Since $A \cap B$ is connected, this intersection is contained in either A_1 or in A_2; renumbering these sets if necessary, we may as well assume that $A \cap B \subset A_1$, which means that $\emptyset = A \cap B \cap A_2 = B \cap A_2$. Since $B \neq \emptyset$ and $B \cap A_2$ is empty, the set $B \cap A_1$ must be nonempty. Consider the closed subspaces $B \cup A_1$ and A_2. They are both nonempty, they are disjoint, and their union is $A \cup B$. Therefore $A \cup B$ is disconnected. On the other hand, we assumed that $A \cup B$ was connected, so we have reached a contradiction. The source of the problem was our added assumption that A was disconnected, so this must be false and we are forced to conclude that A is connected.■

5. Assume that the conclusion is false and that $B \cap \mathrm{Bdy}\,(A) = \emptyset$. Therefore no point of B is a boundary point of A; in other words, for every point $b \in B$ there is some open neighborhood U such that either U does not contain any points of X or else U does not contain any points of $X - A$. This can be rephrased to state that either $U \cap B$ is contained in $B \cap A$ or in $B \cap X - A$ depending upon which of these sets contains b. It follows that $B \cap A$ and $B \cap X - A$ are open subsets in B which are disjoint and whose union is B. By our hypotheses we also know that each intersection is nonempty, and therefore we conclude that B is disconnected. The latter contradicts another of our hypotheses; the source of the contradiction was the added condition that $B \cap \mathrm{Bdy}\,(A) = \emptyset$, and therefore this statement cannot be true. We can rephrase this to say that $B \cap \mathrm{Bdy}\,(A) \neq \emptyset$ must be true.■

5. (*i*) If X is discrete and $x \in X$, then $\{x\}$ by itself forms a neighborhood base at x; since $\{x\}$ is open and closed in X, it follows that X is totally disconnected in the sense of the definition.

As noted in the hint, a discrete space has no limit points because the deleted neighborhood $\{x\} - \{x\}$ is empty. However, the set A of all points x in $\mathbb{R}$ such that $x = 0$ or $x = 1/n$ for some positive integer n does have a limit point at 0, and we claim it is totally disconnected. Since each one point subset $\{1/n\}$ is open and closed, there is a clopen (closed + open) neighborhood base at such points. There is also a clopen neighborhood base at 0, and it is given by the sets

$(-1/n, 1/n) \cap A$ because this set is equal to $[-2/(2n+1), 2/(2n+1)] \cap A$ (the point $1/(n+1)$ is the maximal point in this set). Therefore A is not discrete but A is totally disconnected.

(*ii*) The first set is open and the second set is closed, and the sets in the first family form a neighborhood base, so it is enough to show that the intersections of the open and closed intervals with $\mathbb{Q}$ are equal. Since the closed intervals are obtained from the open intervals by adjoining the two endpoints, it is enough to note that these endpoints are not rational numbers — a consequence of the fact that $\sqrt{2}$ is irrational. This proves that $\mathbb{Q}$ is totally disconnected.

To see that every point of $\mathbb{Q}$ is a limit point of $\mathbb{Q}$, note that if $q \in \mathbb{Q}$ then the sequence of points $q + \frac{1}{n}$ is rational and no point in the sequence is equal to q, but nevertheless the limit of the sequence is q.■

(*iii*) Suppose that X and Y are totally disconnected and $(x, y) \in X \times Y$. Let $\{U_\alpha\}$ and $\{Y_\beta\}$ be clopen neighborhood bases at x and y respectively. Since the product of two clopen subsets is clopen, it follows that $\{U_\alpha \times Y_\beta\}$ forms a clopen neighborhood base at (x, y). Since the latter point was arbitrary, it follows that $X \times Y$ is totally disconnected.■

7. Clearly there are two cases, one when $n = 1$ and the other when $n \geq 2$.

The case $n = 1$. We know that the connected subsets of $\mathbb{R}$ are precisely those sets A such that if $x < y$ and $x, y \in A$, then the closed interval $[x, y]$ is contained in A. Intuitively, we expect that the only sets with this property are open intervals (possibly with $\pm\infty$ as endpoints), half-open intervals (possibly with $\pm\infty$ as the open endpoint), and closed intervals. If this is true, then there are $2^{\aleph_0}$ choices for each endpoint, and for every pair of endpoints there are at most four intervals depending upon which of the ≤ 2 endpoints belong to the interval, so an upper bound for the cardinality of the family of connected subsets is $4 \cdot 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$, This number is also a lower bound because we have the family of intervals $[r, \infty)$ where r runs through the elements of $\mathbb{R}$, and therefore the cardinality of the family of connected subsets is exactly $2^{\aleph_0}$.

We now have to verify the assertion that connected subsets of $\mathbb{R}$ are intervals. Let C be a nonempty connected subset of $\mathbb{R}$, so that $u, v \in C$ and $u < v$ imply $[u, v] \subset C$. Let $a(C)$ be the greatest lower bound of C if C has a lower bound, and let $a(C) = -\infty$ otherwise. Similarly, let $b(C)$ be the least upper bound of C if C has an upper bound, and let $b(C) = +\infty$ otherwise. Let p be some point of C.

CLAIM: The set C is an interval whose lower endpoint is $a(C)$ and whose upper endpoint is $b(C)$. — Since $a(C)$ and $b(C)$ are lower and upper bounds for C, it follows that C is contained in the set of all points x satisfying $a(C) \leq x \leq b(C)$, with the convention that there is strict inequality if $a(C) = -\infty$ or $b(C) = +\infty$. If $a(C) < b(C)$ and $p \in C$ lies between these values, take sequences $\{u_n\}$ and $\{v_n\}$ in C such that $u_n \to a(C)$, $v_n \to b(C)$, and $u_n < p < v_n$ for all n. By the assertion in the first sentence of the first paragraph, then $[u_n; v_n] \subset C$ for all n, and therefore the union of these subsets, which is the set of all x such that $a(C) \leq x \leq b(C)$, is contained in C. Combining this with the previous paragraph, we have

$$\{x \in \mathbb{R} \mid a(C) \ < \ x \ < \ b(C)\} \quad \subset \quad C \quad \subset \quad \{x \in \mathbb{R} \mid a(C) \leq x \leq b(C)\}$$

with the previous conventions if $a(C) = -\infty$ or $b(C) = +\infty$. For each pair of values $a(C)$ and $b(C)$ there are up to four choices for C, depending upon whether or not $a(C)$ or $b(C)$ belong to C.■

The case $n \geq 2$. In order to avoid awkward typographical problems, let $\mathbf{c} = 2^{\aleph_0} = |\mathbb{R}|$; recall that we then have $|\mathbb{R}^n| = \mathbf{c}$ for all positive integers n. Since the set of all subsets in $\mathbb{R}^n$ then has cardinality $2^{\mathbf{c}}$, it follows that $|C_n| \leq 2^{\mathbf{c}}$ for all n. Furthermore, since $\mathbb{R}^2$ is homeomorphic to a subspace of $\mathbb{R}^n$ if $n \geq 3$ it follows that $|C_2| \leq |C_n|$ for all $n \geq 3$. Thus if we can show that $|C_2| \geq 2^{\mathbf{c}}$, then it will follow that $|C_n| = 2^{\mathbf{c}} > \mathbf{c}$ for all $n \geq 2$.

We can construct a family of $2^{\mathbf{c}}$ connected subsets of $\mathbb{R}^2$ as follows: One of our results implies that if A is a connected subset of a topological space and $A \subset B \subset \overline{A}$, then B is connected. In particular, if we apply this to $U = (0,1)^2 \subset \mathbb{R}^2$, then each subset C of $\mathrm{Bdy}\,(U)$ determines a connected subset $U \cup C$ of $\mathbb{R}^2$, and $C \neq C'$ implies that $U \cup C \neq U \cup C'$ because $U \cap \mathrm{Bdy}\,(U) = \emptyset$. In particular, if we let C run through all the $2^{\mathbf{c}}$ subsets of $(0,1) \times \{0\}$, then this yields a family $\{U \cup C\}$ of $2^{\mathbf{c}}$ connected subsets of $\mathbb{R}^2$, and therefore we have proved the desired inequality $|C_2| \geq 2^{\mathbf{c}}$. As noted before, this completes the proof that $|C_n| = 2^{\mathbf{c}} > 2^{\aleph_0}$ if $n \geq 2$.■

8. (*i*) Each line px in $\mathrm{Lines}\,(p, \mathbb{R}^n - D)$ is an arcwise connected set containing p, so every point in the set lies in the same arc component as p, and therefore $\mathrm{Lines}\,(p, \mathbb{R}^n - D)$ is arcwise connected.

The complement of $\mathrm{Lines}\,(p, \mathbb{R}^n - D)$ consists of all points y on punctured lines $px - \{p\}$ such that px contains a point of D; this point in D cannot be p because $p \notin D$. Since there is a unique line joining two points (see the discussion below), every such line px is equal to pz for some $z \in D$. Since there are only countably many points in D, there are only countably many lines px such that $px - \{p\}$ is contained in the complement of $\mathrm{Lines}\,(p, \mathbb{R}^n - D)$.

(*ii*) The existence of a point $x \notin pq$ is discussed in the postscript to the discussion following the solution to this exercise, so the hint involves something which actually exists. If $y \in px$, then $q \notin px$ implies that the lines px and qy are distinct and hence can only have the point y in common (because there is a unique line joining two points). By hypotheses $q \notin D$, and therefore there are only countably many lines of the form qz where $z \in D$, and they are all distinct from px because $q \notin px$ (again by the uniqueness of a line joining two points). Therefore there are only countably many points $y \in px$ such that qy contains an element of D. Since there are uncountably many points on the line px, it follows that there is some point $w \in px - \{p\}$ such that $qw \cap D = \emptyset$. We then have $qw \subset \mathbb{R}^n - D$, and this implies that $pw \cup qw \subset \mathbb{R}^n - D$, which in turn means that p and q lie in the same arc component of $\mathbb{R}^n - D$. Since p and q are arbitrary points of $\mathbb{R}^n - D$, this shows that the latter is arcwise connected.■

Remark on lines in $\mathbb{R}^n$. The solution to this exercise used the following standard geometrical fact, both as intuition and as a logical step in the argument:

Two distinct lines in $\mathbb{R}^n$ have at most one point in common.

We also use a related statement: *Given two points, there is a unique line containing them.*

Since a rigorous proof of the first statement might not have been given in prerequisite undergraduate courses, for the sake of completeness we shall give a self-contained proof here using vector algebra. One can also use the concepts developed in Sections I.3 and II.2 of `pg-all.pdf` to view this result as part of a more general pattern. We should probably start by formally defining the **xy** joining two points $\mathbf{x}$, $\mathbf{y} in \mathbb{R}^n$ (or more generally vectors in any vector space V) to be the set of all points $\mathbf{z}$ such that $\mathbf{z} = t\,\mathbf{y} + (1-t)\,\mathbf{x} = \mathbf{x} + t\,(\mathbf{y} - \mathbf{x})$ for some scalar t. It is an elementary exercise in linear algebra to check that a point $\mathbf{w}$ belongs to this set if and only if $\mathbf{w} = \mathbf{y} + s\,(\mathbf{y} - \mathbf{x})$ for some scalar s (in fact, if $\mathbf{w} = \mathbf{z}$ where $\mathbf{z}$ is given as above then we can take $s = 1 - t$).

Assume now that we are given $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{c} \neq \mathbf{d}$ in $\mathbb{R}^n$ such that the lines **ab** and **cd** have two distinct points in common. We need to prove that **ab** must be equal to **cd**.

If the lines have two points in common, then there are scalars s, t, u, v such that $t\,\mathbf{b} + (1-t)\,\mathbf{a} = s\,\mathbf{d} + (1-s)\,\mathbf{c}$ and $v\,\mathbf{b} + (1-v)\,\mathbf{a} = u\,\mathbf{d} + (1-u)\,\mathbf{c}$. We may rewrite these in the forms

$$\mathbf{a} \;+\; t\,(\mathbf{b} - \mathbf{a}) \;\;=\;\; \mathbf{c} \;+\; s\,(\mathbf{d} - \mathbf{c})\;, \quad \mathbf{a} \;+\; v\,(\mathbf{b} - \mathbf{a}) \;\;=\;\; \mathbf{c} \;+ u\,(\mathbf{d} - \mathbf{c})$$

and if we take the difference of these equations we find that $(t-v)\,(\mathbf{b}-\mathbf{a}) = (s-u)\,(\mathbf{d}-\mathbf{c})$. Therefore there is a nonzero vector $\mathbf{w}$ such that $\mathbf{b}-\mathbf{a} = K\,\mathbf{w}$ and $\mathbf{d}-\mathbf{c} = L\,\mathbf{w}$; since the expressions on the left are nonzero, it follows that K and L are both nonzero. These yield the equation

$$\mathbf{c}-\mathbf{a} \quad = \quad K\,(t-v)\,\mathbf{w} \quad = \quad L\,(s-u)\,\mathbf{w}$$

so that $K\,(t-v) = L\,(s-u)$. Therefore for some scalars M and M' we have

$$\mathbf{c} \quad = \quad \mathbf{a} \;+\; M\,\mathbf{w} \quad = \quad \mathbf{a} \;+\; M'\,(\mathbf{b}-\mathbf{a})$$

so that $\mathbf{c} \in \mathbf{ab}$. If we now reverse the roles of $\mathbf{c}$ and $\mathbf{d}$ in the preceding argument, we also find that $\mathbf{d} \in \mathbf{ab}$.

We shall now prove that $\mathbf{cd} \subset \mathbf{ab}$; if we know this, then we can reverse the roles of the two lines in the argument and also obtain the conclusion $\mathbf{ab} \subset \mathbf{cd}$, and if we combine these we have $\mathbf{ab} = \mathbf{cd}$, which is what we wanted to prove. — To see this, let $\mathbf{x} in \mathbf{cd}$. Then $\mathbf{x} = r\,\mathbf{c} + (1-r)\,\mathbf{d}$ for some r. Now $\mathbf{c}, \mathbf{d} \in \mathbf{ab}$ implies that $\mathbf{c} = (1-M')\,\mathbf{a} + M'\,\mathbf{b}$ (as before) and $\mathbf{d} = (1-N')\,\mathbf{a} + N'\,\mathbf{b}$ for some scalars M' and N'. Direct substitution then yields the equation

$$\mathbf{x} \quad = \quad (1 - rM' + (r-1)N')\,\mathbf{a} + (rM' + (1-r)N')\,\mathbf{b}$$

which shows that $\mathbf{ab} \subset \mathbf{cd}$. As noted earlier in this paragraph, similar reasoning shows the reverse inclusion, and therefore the two lines must be equal.■

Postscript. In our solution we also use the fact that if $n \geq 2$ and $\mathbf{p} \neq \mathbf{q} \in \mathbb{R}^n$, then there is some point $\mathbf{x}$ which does not lie on the line $\mathbf{pq}$, so to be complete we should also verify this assertion. By definition, if $\mathbf{x}$ lies on $\mathbf{pq}$ then $\mathbf{x}-\mathbf{p}$ is a scalar multiple of $\mathbf{q}-\mathbf{p}$. Since $n \geq 2$ implies the existence of some vector $\mathbf{v}$ which is not a multiple of $\mathbf{q}-\mathbf{p}$, it follows that $\mathbf{x} = \mathbf{p} + \mathbf{v}$ does not lie on $\mathbf{pq}$.■

III.5 : Variants of connectedness

Problem from Munkres, § 24, pp. 157 – 159

8. (b) The closure $\overline{A}$ of an arc component A is NOT NECESSARILY arcwise connected; this contrasts with the fact that connected components are always closed subsets. Consider the graph of $\sin(1/x)$ for $x > 0$. It closure is obtained by adding the points in $\{0\} \times [-1, 1]$ and we have shown that the space consisting of the graph and this closed segment is not path connected.■

Problem from Munkres, § 25, pp. 162 – 163

10. [Only (a), (b) and examples A and B from (c).]

(a) The proofs that $\sim$ is reflexive and symmetric are very elementary and left to the reader. Regarding transitivity, suppose that $x \sim y$ and $y \sim z$ and that we are given a separation of X as $A \cup B$. Without loss of generality we may assume that $x \in A$ (otherwise interchange the roles of A and B in the argument). Since $x \sim y$ it follows that $y \in A$, and the latter combines with $y \sim z$ to show that $z \in A$. Therefore if we are given a separation of X as above, then x and z lie in the same piece, and since this happens for all separations it follows that $x \sim z$.■

(b) Let C be a connected component of X. Then $A \cap C$ is an open and closed subset of C and therefore it is either empty or all of C. In the first case $C \subset B$ and in the second case $C \cap B$. In either case it follows that all points of C lie in the same equivalence class. – If X is locally connected then each connected component is both closed and open. Therefore, if x and y lie in different components, say C_x and C_y, then $X = C_x \cup X - C_x$ defines a separation such that x lies in the first subset and y lies in the second, so that $x \sim y$ is false if x and y do not lie in the same connected component. Combining this with the previous part of the exercise, if X is locally connected then $x \sim y$ if and only if x and y lie in the same connected component.■

(c) [*Note:* Example C in this part of the problem was not assigned; some comments appear below.]

> *Trying to solve this exercise without any pictures would probably be difficult at best and hopelessly impossible at worst.*

Drawings for Case A and Case C are in the file `math205Axolutions03a.pdf`.

Case A**.** We claim that the connected components are the closed segments $\{1/n\} \times [0,1]$ and the one point subsets $\{(0,0)\}$ and $\{(0,1)\}$. — Each segment is connected and compact (hence closed), and we claim each is also an open subset of A. This follows because the segment $\{1/n\}\times[0,1]$ is the intersection of A with the open subset

$$\left(\frac{1}{2}\left[\frac{1}{n} + \frac{1}{n+1}\right] \ , \ \frac{1}{2}\left[\frac{1}{n} + \frac{1}{n-1}\right] \right) \times \mathbb{R} \ .$$

Since the segment $\{1/n\} \times [0,1]$ is open, closed and connected, it follows that it must be both a component and a quasicomponent. It is also an arc component because it is arcwise connected. This leaves us with the two points $(0,0)$ and $(0,1)$. They cannot belong to the same component because they do not form a connected set. Therefore each belongs to a separate component and also to a separate path component. The only remaining question iw whether or not they determine the same quasicomponent. To show that they do lie in the same quasicomponent, it suffices to check that if $A = U \cup V$ is a separation of X into disjoint open subsets then both points lie in the same open set. Without loss of generality we may as well assume that the origin lies in U. It then follows that for all n sufficiently large the points $(1/n, 0)$ all lie in U, and the latter implies that all of the connected segments $\{1/n\} \times [0,1]$ is also lie in U for n sufficiently large. Since $(0,1)$ is a limit point of the union of all these segments (how?), it follows that $(0,1)$ also lies in U. This implies that $(0,1)$ lies in the same quasicomponent of A as $(0,0)$.■

Case B**.** This set turns out to be connected but not arcwise connected. We claim that the path components are given by $\{(0,1)\}$ and its complement. Here is the proof that the complement is path connected: Let P be the path component containing all points of $[0,1] \times \{0\}$. Since the latter has a nontrivial intersection with each vertical closed segment $\{1/n\} \times [0,1]$ it follows that all of these segments are also contained in P, and hence P consists of all points of A except perhaps $(0,1)$. Since $(0,1)$ is a limit point of P (as before) it follows that B is connected and thus there is only one component and one quasicomponent. We claim that there are two path components. Suppose the extra point $(0,1)$ also lies in P. Then, for example, there will be a continuous curve $\alpha : [0,1] \to A$ joining $(0,1)$ to $(1/2,1)$. Let t_0 be the maximum point in the subset of $[0,1]$ where the first coordinate is zero. Since the first coordinate of $\alpha(a)$ is 1, we must have $t_0 < 1$. Since $A \cap \{0\} \times \mathbb{R}$ is equal to $\{(0,0)\} \cup \{(0,1)\}$, it follows that α is constant on $[0, t_0]$, and by continuity there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies that the *second* coordinate of $\alpha(t)$ is greater than $1/2$. If $t \in (t_0, t_0 + (\delta/2))$, then the first coordinate of $\alpha(t)$ is positive and the second is greater than $1/2$. In fact we can choose some t_1 in the open interval such that the first coordinate is irrational. But there are no points in $B \cap (\frac{1}{2}, +\infty)$ whose first coordinates are irrational, so we have a contradiction. The latter arises because we assumed the existence of a continuous curve joining $(0,1)$ to another point in B. Therefore no such curve can exist and $(0,1)$ does not belong to the path component P. Hence B has two path components, and one of them contains only one point.■

Case C. This was not assigned, but we shall include the solution as an example of an argument which is more challenging and should be understood passively. Before proceeding to the solution, we shall give away the answer: The space C is connected and each of the (infinitely many) closed segments given in the definition is a separate path component.

Once again, it would probably be extremely difficult to solve this part of the exercise without the sort of drawing we have inserted into `math205Asolutions03a.pdf`. This drawing shows that C is symmetric with respect to rotation through $90°$;*i.e.*, if $J(x,y) = (-y,x)$ is counterclockwise rotation through a right angle, then $J[C] = C$. As usual, one needs to give a written argument to verify this, but this is routine and the details are left to the reader (note that J maps the first summand of C to the fourth, the fourth summand to the second, the second summand to the third, and the third summand to the first).

The set C is a union of countably closed segments $E_{j,n}$ which are defined by $E_{0,n} = \{1/n\} \times [0,1]$ and $E_{k,n} = J^k[E_{0,n}] = J^{k-1}[E_{k-1,n}]$, where J is defined as in the preceding paragraph. Set $E_k = \cup_n \; E_{k,n}$. — Also, let $K = \{1, \frac{1}{2}, \frac{1}{3}, \; \cdots \}$ be defined as in Munkres' formulation of the exercise.

Suppose that D is a clopen subset of C which contains $(1,0)$. Since D is open, it also contains points of the form $(1/n, 1)$ for all but finitely many values of n, and hence it contains points from all but finitely many of the intervals $E_{3,n}$. By the connectedness of these intervals and the clopenness of D, it follows that $E_{3,n} \subset D$ for $n \geq N_0$ for some N_0. Each of the points $(1/m, 0)$ is a limit point of $\cup_{n \geq N_0} \; E_{3,n}$ (consider the sequence $(1/m, 1/n)$ for $n \geq N_0$), and since D is closed it also follows that $(1/m, 0) \in D$. Since the latter point lies in $E_{0,m}$ and the latter is connected, it follows that the clopen set D also contains E_0. Furthermore, since $(0,1)$ is a limit point of E_0 (look at the sequence $(1/m, 1)$ in E_0), it follows that $J(1,0) = (0,1) \in D$.

By the same reasoning, if $k \geq 1$ we know that if $J^k(0,1) \in D$ then $E_j \subset D$ and $J^{k+1}(0,1) \in D$. Therefore we can put together an inductive argument to conclude that $E_k \subset D$ for all k (since J^4 is the identity, there are only finitely many steps where a new conclusion is obtained). Finally, since $C = \cup_j \; E_j$ it follows that $C = D$, which means that C is connected. Therefore there is only one connected component, and there is also only one quasicomponent because each quasicomponent is a union of components.

We must now describe the arc components of C; as noted earlier, we want to show that these are the subspaces $E_{k,n}$. Most of the work involves proving the following:

> CLAIM . *If $x \in E_{k,n}$ and U is an open neighborhood of x in C, then there is a subneighborhood V of U such that the arc component of x in V is equal to $E_{k,n} \cap V$.*

Because of the symmetry of C under right angle rotations, it will suffice to prove this claim when $k = 0$.

To simplify the notation, let $Sq_\varepsilon(x) \subset \mathbb{R}^2$ denote the open square of all points $y \in \mathbb{R}^2$ such that $\mathbf{d}^{\langle\infty\rangle}(x,y) < \varepsilon$. We shall show that if $x \in C$ as above then we can take V to have the form $Sq_\varepsilon(x) \cap C$ for some $\varepsilon > 0$ (which suffices since the latter subsets form a neighborhood base at x).

As above, we need only consider the case where $x \in E_{0,n_0}$ for some positive integer n_0, so we have $x = (1/n_0, t_0)$ for some $t_0 \in [0,1]$. If $t_0 > 0$ then $V = Sq_\varepsilon(x) \cap C$ has the desired properties if we take ε to be smaller than both t_0 and $1/(2n_0)$; in fact, for this case the intersection lies on the vertical line through x and it is arcwise connected. We now turn to the more complicated case where $t_0 = 0$. In this case, if $\varepsilon < 1/(2n_0)$ then $V = Sq_\varepsilon(x) \cap C$ is a disjoint union of the vertical segment $\{1/n_0\} \times [0, \varepsilon)$ and the horizontal segments $(1/n_0 - \varepsilon, 1/n_0 + \varepsilon) \times \{1/m\}$ for all integers m such that $1/m < \varepsilon$. From this description it is clear that the arc component of x in V is the intersection of the latter with the vertical segment E_{0,n_0}. This completes the proof of the claim (for the sets $E_{0,n}$; as noted before, symmetry considerations yield the same conclusion for $E_{k,n}$ when $k = 1, 2, 3$).

Using the claim we have just verified, we can prove that each subset $E_{k,n}$ is an arc component as follows: Let $\gamma : [0,1] \to C$ be a continuous curve. If $\gamma(t) \in E_{k,n}$ then the claim and the continuity of γ imply that γ maps some interval $(t-h, t+h) \cap [0,1]$ into $E_{0,n}$, and this implies that $\gamma^{-1}[E_{k,n}]$ is an open subset of $[0,1]$ for each k and n. Since the sets $E_{k,n}$ are pairwise disjoint and their union is C, it follows that the open subsets $\gamma^{-1}[E_{k,n}] \subset [0,1]$ are also pairwise disjoint and their union is $[0,1]$. Since the unit interval is connected, it follows that exactly one of these subsets is nonempty, and therefore it must be the entire unit interval. In other words, we have shown that the image of a continuous mapping from $[0,1]$ to C must be contained in some $E_{k,n}$, which means that every arc component of C must be contained in some $E_{k,n}$. Finally, since the latter are arcwise connected, it follows that each $E_{k,n}$ is an arc component of C.■

Additional exercises

1. In a locally connected space the connected components are open (and pairwise disjoint). These sets form an open covering and by compactness there is a finite subcovering. Since no proper subcollection of the set of components is an open covering, this implies that the set of components must be finite.■

2. Let $Y = \mathbb{R}^2$ and let $X \subset \mathbb{R}^2$ be the union of the horizontal half-line $(0, \infty) \times \{0\}$ and the vertical closed segment $\{-1\} \times [-1, 1]$. These subsets of X are closed in X and pairwise disjoint. Let $f : X \to \mathbb{R}^2$ be the continuous map defined on $(0, \infty) \times \{0\}$ by the formula $f(t, 0) = (t, \sin(1/t))$ and on $\{-1\} \times [-1, 1]$ by the formula $f(-1, s) = (0, s)$. The image $f[X]$ is then the example of a non-locally connected space that is described in the course notes.■

3. Consider the polar coordinate map $[1, \sqrt{2}] \times [0, 2\pi]$ which sends (r, θ) to $(r\cos\theta, r\sin\theta)$. This is a continuous onto mapping, and the domain is arcwise connected (in fact, it is a convex subset of $\mathbb{R}^2$. Therfore the image, which is the annulus in the exercise, is also arcwise connected.■

4. The result for $\mathbb{R}^n - \{\mathbf{0}\}$ is implicit in one of the earlier starred exercises, but we shall give a proof here because it can be done simply and the result for S^{n-1} depends upon this fact.

If $x, y \in \mathbb{R}^n - \{\mathbf{0}\}$ are linearly independent, then the line segment $ty + (1-t)x$ $(0 \leq t \leq 1)$ does not pass through $\mathbf{0}$ (if it did, then x and y would be linearly dependent), and hence x and y lie in the same arc component of $\mathbb{R}^n - \{\mathbf{0}\}$. Suppose now that x and y are linearly dependent, so that each is a nonzero multiple of the other. Since $n \geq 2$ there is some vector z such that x and z are linearly independent, and it follows immediately that y and z are also linearly independent. Two applications of the previous argument then show that x, y, z all lie in the same arc component. Thus in all cases we have shown that two arbitrary points $x, y \in \mathbb{R}^n - \{\mathbf{0}\}$ always lie in the same arc component, which means that the space under consideration is arcwise connected.

To prove the result for S^{n-1}, consider the map

$$\sigma : \mathbb{R}^n - \{\mathbf{0}\} \longrightarrow S^{n-1}$$

which sends a nonzero vector v to the unit vector $|v|^{-1} \cdot v$ pointing in the same direction. This map is continuous since $|v| \neq 0$ on the domain, and it is clearly onto because $\sigma(v) = v$ if $v \in S^{n-1}$ (so that $|v| = 1$). Since $\mathbb{R}^n - \{\mathbf{0}\}$ is arcwise connected, its image under σ — which is S^{n-1} — must also be arcwise connected.■

5. (*i*) Let Q denote the quasicomponent of p in X. By definition $q \in Q$ if and only if for each separation of X into clopen subsets $A \cup B$ (with $A \cap B = \emptyset$) both p and q lie in the same subset.

By definition, if $q \in Q$ and $C \subset X$ is a clopen subset containing p, then $q \in C$. Therefore Q is contained in the intersection of all clopen subsets C such that $p \in C$. Conversely, if q lies in this intersection, then if C is a clopen set containing p we have $q \in C$, which means that q lies in the same quasicomponent as p.■

(*ii*) We are assuming there are only finitely many components, so list them as $C_1 \ \cdots \ , C_r$. Each of these subsets is closed since it is a component, and we claim that each is also open. This is true because we have

$$X \ - \ C_j \ = \ \bigcup_{i \neq j} C_i$$

for all j, so that the right hand side is a closed subset and hence its complement — which is C_j — must be open. This means that every component of X is clopen, and by (*i*) it follows that the quasicomponent of a point is contained in the component of a point. On the other hand, by Exercise 25.10 in Munkres we know that the reverse inclusion is true, and therefore the quasicomponent of a point is equal to the connected component of that point.■