SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 4

Fall 2014

IV. Function spaces

IV.1 : General properties

Additional exercises

1. The mapping q is 1–1 because $q(f) = q(g)$ implies that for all x we have $f(x) = p_x \circ q(f) = p_x \circ q(g) = g(x)$, which means that $f = g$.

To prove continuity, we need to show that the inverse images of subbasic open sets in Y^X are open in $\mathbf{C}(X,Y)$. The standard subbasic open subsets have the form $\mathcal{W}(\{x\},U) = p_x^{-1}(U)$ where $x \in X$ and U is open in Y. In fact, there is a smaller subbasis consisting of all such sets $\mathcal{W}(\{x\},U)$ such that $U = N_\varepsilon(y)$ for some $y \in Y$ and $\varepsilon > 0$. Suppose that f is a continuous function such that $q(f)$ lies in $\mathcal{W}(\{x\},U)$. By definition the later condition means that $f(x) \in U$. The latter in turn implies that $\delta = \varepsilon - \mathbf{d}(\,f(x), y) > 0$, and if $\mathbf{d}(f,g) < \delta$ then the Triangle Inequality implies that $\mathbf{d}(\,g(x), y) < \varepsilon$, which in turn means that $g(x) \in U$. Therefore q is continuous at f, and since f is arbitrary this shows q is a continuous mapping.■

2. It suffices to show that the map in question is onto and distance-preserving. The map is onto because if u and v are continuous functions into Y and Z respectively, then we can retrieve f by the formula $f(x) = \big(\,u(x),\, v(x)\,\big)$. Suppose now that f and g are continuous functions from X to $Y \times Z$. Then the distance from f to g is the maximum of $\mathbf{d}\big(\,f(x),\, g(x)\,\big)$. The latter is less than or equal to the greater of $\mathbf{d}\big(\,p_Y f(s),\, p_Y g(s)\,\big)$ and $\mathbf{d}\big(\,p_Y f(t),\, p_Y g(t)\,\big)$. Thus if

$$\Phi : \mathbf{C}(X, Y \times Z) \to \mathbf{C}(X,Y) \times \mathbf{C}(X,Z)$$

then the distance from $\Phi(f)$ to $\Phi(g)$ is greater than or equal to the distance from f to g. This means that the map Φ^{-1} is uniformly continuous. Conversely, we claim that the distance from f to g is greater than or equal to the distance between $\Phi(f)$ and $\Phi(g)$. The latter is equal to the larger of the maximum values of $\mathbf{d}\big(\,p_Y \circ f(s),\, p_Y \circ g(s)\,\big)$ and $\mathbf{d}\big(\,p_Z \circ f(t),\, p_Z \circ g(t)\,\big)$. If $w \in X$ is where $\mathbf{d}(f,g)$ takes its maximum, it follows that

$$\mathbf{d}\big(\,f(w),\, g(t)\,\big) \quad =$$

$$\max\Big\{\ \mathbf{d}\big(\,p_Y \circ f(w),\, p_Y \circ g(w)\,\big),\ \mathbf{d}\big(\,p_Z \circ f(w),\, p_Z \circ g(w)\,\big)\ \Big\}$$

which is less than or equal to the distance between $\Phi(f)$ and $\Phi(g)$ as described above.■

3. The distance between f and g is the maximum value of the distance between $f(x)$ and $g(x)$ as x runs through the elements of x, which is the greater of the maximum distances between $f(x)$ and $g(x)$ as x runs through the elements of C, where C runs through the set $\{A, B\}$. But the second expression is equal to the larger of the distances between $\mathbf{d}(f|A, g|A)$ and $\mathbf{d}(f|B, g|B)$.

Therefore the map described in the problem is distance preserving. As in the previous exercise, to complete the proof it will suffice to verify that the map is onto. The surjectivity is equivalent to saying that a function is continuous if its restrictions to the closed subsets A and B are continuous. But we know the latter is true.$\blacksquare$

4. Let $\varepsilon > 0$ be given. We claim that the distance between $f \times g$ and $f' \times g'$ is less than ε if the distance between f and f' is less than ε and the distance between g and g' is less than ε. Choose $u_0 \in X$ and $v_0 \in Z$ so that

$$\mathbf{d}\big(\, f' \times g'(u_0, v_0),\ f \times g(u_0, v_0),\, \big)$$

is maximal and hence equal to the distance between $f \times g$ and $f' \times g'$. The displayed quantity is equal to the greater of $\mathbf{d}\big(\, f'(u_0),\, f(u_0)\,\big)$ and $\mathbf{d}\big(\, g'(v_0),\, g(v_0)\,\big)$. These quantities in turn are less than or equal to $\mathbf{d}(f', f)$ and $\mathbf{d}(g', g)$ respectively. Therefore if both of the latter are less than ε it follows that the distance between $f' \times g'$ and $f \times g$ is less than ε.$\blacksquare$

5. (*i*) Follow the hint. We then have $V(h) \circ V(f) = V(h \circ f) = V(\mathrm{id})$, which is the identity. Likewise, we also have $V(f) \circ V(h) = V(f \circ h) = V(\mathrm{id})$, which is the identity.$\blacksquare$

(*ii*) Again follow the hint. We then have $U(f) \circ U(h) = U(h \circ f) = V(\mathrm{id})$, which is the identity. Likewise, we also have $U(h) \circ V(f) = V(f \circ h) = V(\mathrm{id})$, which is the identity.$\blacksquare$

(*iii*) Let $f : A \to A'$ and $g : B \to B'$ be the homeomorphisms. Let $U(f^{-1}) \circ V(g) = V(g) \circ U(f^{-1})$ — equality holds by associativity of composition. By the first two parts of the exercise this map is the required homeomorphism from $\mathbf{C}(A, B)$ to $\mathbf{C}(A', B')$.$\blacksquare$

6. It will be convenient to denote $\alpha(\gamma, \gamma')$ generically by $\gamma + \gamma'$. The construction then implies that the distance between $\xi + \xi'$ and $\eta + \eta'$ is the larger $\mathbf{d}(\xi, \eta)$ and $\mathbf{d}(\xi', \eta')$. Therefore the concatenation map is distance preserving.

7. Follow the hint. If $y \neq y'$, then the values of $k(y)$ at every point is X, and hence it is not equal to y', the value of $k(y')$ at every point of X. Therefore k is 1–1.

Next, we shall verify the set-theoretic identities described above. If $y \in U$ then since $k(y)$ is the function whose value is y at every point we clearly have $k(y) \in \mathcal{W}(K, U)$, and hence $k(U) \subset \mathcal{W}(K, U) \cap \mathrm{Image}(k)$. Conversely, any constant function in the image of the latter is equal to $k(y)$ for some $y \in U$. The identity $k^{-1}(\,\mathcal{W}(K, U)\,) = U$ follows similarly.$\blacksquare$

8. As indicated in the hint, without loss of generality we may assume that $s < t$. Given a function f in $\mathbf{Diff}(A, B)$, the Mean Value Theorem implies that

$$f(t) \ - \ f(s) \quad = \quad f'(\xi) \cdot (t - s)$$

for some $\xi \in (s, t)$. By hypothesis $|f'(x)| \leq B$ for all x, and therefore $|f(t) - f(s)| = |f'(\xi)||t - s| \leq B \cdot |t - s|$. Therefore, if $\varepsilon > 0$ and $\delta = \varepsilon / B$, then $|s - t| < \delta$ implies $|f(s) - f(t)| < \varepsilon$.$\blacksquare$

IV.2 : Adjoint equivalences

Additional exercises

1. The map $A : \mathbf{F}(X \times Y, Z) \to \mathbf{F}(X, \mathbf{F}(Y, Z)\,)$ sends $h : X \times Y \to Z$ to the function $h^\flat$ such that $[h^\flat(x)](y) = h(x, y)$. The argument proving the adjoint formula for spaces of continuous functions modifies easily to cover these examples, and in fact in this case the proof is a bit easier because it is not necessary to consider metrics or topologies.

2. Let $y_0 \in Y$, and let $L : Y \times [0,1] \to Y$ be the map sending (y,t) to the point $(1-t)y + ty_0$ on the line segment joining y to y_0. If we set $H(x,t) = L(\,f(x), t)$, then H satisfies the conditions in the hint and defines a continuous map in $\mathbf{C}(X,Y)$ joining f to the constant function whose values is always y_0. Thus for each f we know that f and the constant function with value y_0 lie in the same arc component of $\mathbf{C}(X,Y)$. Therefore there must be only one arc component.■

3. By the adjoint formula there are homeomorphisms

$$\mathbf{C}\big(X,\,\mathbf{C}(Y,Z)\big) \;\cong\; \mathbf{C}(X \times Y, Z) \;\cong\; \mathbf{C}(Y \times X, Z) \;\cong\; \mathbf{C}\big(Y,\,\mathbf{C}(X,Z)\big)$$

and this yields the desired 1–1 correspondence of sets.■

V. Constructions on spaces

V.1 : Quotient spaces

Problem from Munkres, § 22, pp. 144 – 145

4. (a) The hint describes a well-defined continuous map from the quotient space W to the real numbers. The equivalence classes are simply the curves $g(x, y) = C$ for various values of C, and they are parabolas that open to the left and whose axes of symmetry are the x-axis. It follows that there is a 1–1 onto continuous map from W to $\mathbb{R}$. How do we show it has a continuous inverse? The trick is to find a continuous map in the other direction. Specifically, this can be done by composing the inclusion of $\mathbb{R}$ in $\mathbb{R}^2$ as the x-axis with the quotient projection from $\mathbb{R}^2$ to W. This gives the set-theoretic inverse to $\mathbb{R}^2 \to W$ and by construction it is continuous. Therefore the quotient space is homeomorphic to $\mathbb{R}$ with the usual topology.■

(b) Here we define $g(x, y) = x^2 + y^2$ and the equivalence classes are the circles $g(x, y) = C$ for $C > 0$ along with the origin. In this case we have a continuous 1–1 onto map from the quotient space V to the nonnegative real numbers, which we denote by $[0, \infty)$ as usual. To verify that this map is a homeomorphism, consider the map from $[0, \infty)$ to V given by composing the standard inclusion of the former as part of the x-axis with the quotient map $\mathbb{R}^2 \to V$. This is a set-theoretic inverse to the map from V to $[0, \infty)$ and by construction it is continuous.■

Additional exercises

0. We claim that every subset of $X/\mathcal{R}$ is both open and closed. But a subset of the quotient is open and closed if and only if the inverse image has these properties, and every subset of a discrete space has these properties.■

1. We need to show that $U \subset A$ is open if and only if $r^{-1}[U]$ is open in X. The ($\Longrightarrow$) implication is immediate from the continuity of r. To prove the other direction, note that $r \circ i = \mathrm{id}_A$ implies that

$$U \quad = \quad i^{-1}\left[r^{-1}[U]\right] \quad = \quad A \cap r^{-1}[U]$$

and thus if the inverse image of U is open in X then U must be open in A.■

2. Let p_X and p_A denote the quotient space projections for X and A respectively. By construction, j is the unique function such that $j \circ p_A = p_X|A$ and therefore j is continuous. We shall define an explicit continuous inverse $k : X/\mathcal{R} \to A/\mathcal{R}_0$. To define the latter, consider the continuous map $p_A \circ r : X \to A/\mathcal{R}_0$. If $y\mathcal{R}z$ holds then $r(y)\mathcal{R}_0\, r(z)$, and therefore the images of y and z in $A/\mathcal{R}_0$ are equal. Therefore there is a unique continuous map k of quotient spaces such that $k \circ p_X = p_A \circ r$. This map is a set-theoretic inverse to j and therefore j is a homeomorphism.■

3. (a) The relation is reflexive because $x = 1 \cdot x$, and it is reflexive because $y = \alpha x$ for some $\alpha \neq 0$ implies $x = \alpha^{-1}y$. The relation is transitive because $y = \alpha x$ for $\alpha \neq 0$ and $z = \beta y$ for $y \neq 0$ implies $z = \beta\alpha x$, and $\beta\alpha \neq 0$ because the product of nonzero real numbers is nonzero.■

(b) Use the hint to define r; we may apply the preceding exercise if we can show that for each $a \in S^2$ the set $r^{-1}(\{a\})$ is contained in an $\mathcal{R}$-equivalence class. By construction $r(v) = |v|^{-1}v$, so $r(x) = a$ if and only if x is a positive multiple of a (if $x = \rho\, a$ then $|x| = \rho$ and $r(x) = a$, while if

$a = r(x)$ then by definition a and x are positive multiples of each other). Therefore if $x\mathcal{R}y$ then $r(x) = \pm\, r(y)$, so that $r(x)\mathcal{R}_0\, r(y)$ and the map

$$S^2/[x \equiv \pm x] \quad \longrightarrow \quad \mathbb{RP}^2$$

is a homeomorphism.■

4. Needless to say we shall follow the hints in a step by step manner.

Let $h : D^2 \to S^2$ be defined by

$$h(x, y) = (x, y, \sqrt{1 - x^2 - y^2}\,).$$

Verify that h preserves equivalence classes and therefore induces a continuous map $\overline{h}$ on quotient spaces.

To show that $\overline{h}$ is well-defined it is only necessary to show that its values on the $\mathcal{R}'$-equivalence classes with two elements are the same for both representatives. If $\pi : S^2 \to \mathbb{RP}^2$ is the quotient projection, this means that we need $\pi \circ h(u) = \pi \circ h(v)$ if $|u| = |v| = 1$ and $u = -v$. This is immediate from the definition of the equivalence relation on S^2 and the fact that $h(w) = w$ if $|w| = 1$.

Why is $\overline{h}$ a $1-1$ and onto mapping?

By construction h maps the equivalence classes of points on the unit circle onto the points of S^2 with $z = 0$ in a 1–1 onto fashion. On the other hand, if u and v are distinct points that are not on the unit circle, then $h(u)$ cannot be equal to $\pm\, h(v)$. The inequality $h(u) \neq -h(v)$ follows because the first point has a positive z-coordinate while the second has a negative z-coordinate. The other inequality $h(u) \neq h(v)$ follows because the projections of these points onto the first two coordinates are u and v respectively. This shows that $\overline{h}$ is 1–1. To see that it is onto, recall that we already know this if the third coordinate is zero. But every point on S^2 with nonzero third coordinate is equivalent to one with positive third coordinate, and if $(x, y, z) \in S^2$ with $z > 0$ then simple algebra shows that the point is equal to $h(x, y)$.

Finally, prove that $\mathbb{RP}^2$ is Hausdorff and $\overline{h}$ is a closed mapping.

If the first statement is true, then the second one follows because the domain of $\overline{h}$ is a quotient space of a compact space and continuous maps from compact spaces to Hausdorff spaces are always closed. Since $\overline{h}$ is already known to be continuous, 1–1 and onto, this will prove that it is a homeomorphism.

So how do we prove that $\mathbb{RP}^2$ is Hausdorff? Let v and w be points of S^2 whose images in $\mathbb{RP}^2$ are distinct, and let P_v and P_w be their orthogonal complements in $\mathbb{R}^3$ (hence each is a 2-dimensional vector subspace and a closed subset). Since Euclidean spaces are Hausdorff, we can find an $\varepsilon > 0$ such that $N_\varepsilon(v) \cap P_v = \emptyset$, $N_\varepsilon(w) \cap P_w = \emptyset$, $N_\varepsilon(v) \cap N_\varepsilon(w) = \emptyset$, and $N_\varepsilon(-v) \cap N_\varepsilon(w) = \emptyset$. If T denotes multiplication by -1 on $\mathbb{R}^3$, then these conditions imply that the four open sets

$$N_\varepsilon(v), \quad N_\varepsilon(w), \quad N_\varepsilon(-v) = T\left(N_\varepsilon(v)\right), \quad N_\varepsilon(-w) = T\left(N_\varepsilon(w)\right)$$

are pairwise disjoint. This implies that the images of the distinct points $\pi(v)$ and $\pi(w)$ in $\mathbb{RP}^2$ lie in the disjoint subsets $\pi\left[\,N_\varepsilon(v)\,\right]$ and $\pi\left[\,N_\varepsilon(w)\,\right]$ respectively. These are open subsets in $\mathbb{RP}^2$ because their inverse images are given by the open sets $N_\varepsilon(v) \cup N_\varepsilon(-v)$ and $N_\varepsilon(w) \cup N_\varepsilon(-w)$ respectively.■

5. A set W belongs to $(g \circ f)_*\mathbf{T}$ if and only if $(g \circ f)^{-1}[W]$ is open in X. But

$$(g \circ f)^{-1}[W] = f^{-1}\left[\,g^{-1}[W]\,\right]$$

so the condition on W holds if and only if $g^{-1}[W]$ belongs to $f_*\mathbf{T}$. The latter in turn holds if and only if w belongs to $g_*\,(f_*\mathbf{T})$.■

6. The object on the left hand side is the family of all sets having the form $(f\circ h)^{-1}[V]$ where V belongs to $\mathbf{T}$. As in the preceding exercise we have

$$(f\circ h)^{-1}[V] = h^{-1}\left[\,f^{-1}[V]\,\right]$$

so the family in question is just $h^*\,(f^*\mathbf{T})$.■

7. Let $p : X \times Y \to X$ be projection onto the first coordinate. Then $u\mathcal{R}v$ implies $p(u) = p(v)$ and therefore there is a unique continuous map $X \times Y/\mathcal{R} \to X$ sending the equivalence class of (x, y) to x. Set-theoretic considerations imply this map is 1–1 and onto, and it is a homeomorphism because p is an open mapping.■

8. (a) If $X/\mathcal{R}$ is Hausdorff then the diagonal $\Delta(X/\mathcal{R})$ is a closed subset of $(X/\mathcal{R}) \times (X/\mathcal{R})$. But $\pi \times \pi$ is continuous, and therefore the inverse image of $\Delta(X/\mathcal{R})$ must be a closed subset of $X \times X$. But this set is simply the graph of $\mathcal{R}$.■

(b) If π is open then so is $pi \times \pi$, for the openness of π implies that $\pi \times \pi$ takes basic open subsets of $X \times X$ into open subsets of $(X/\mathcal{R}) \times (X/\mathcal{R})$. By hypothesis the complementary set $X \times X - \Gamma_{\mathcal{R}}$ is open in $X \times X$, and therefore its image, which is

$$(X/\mathcal{R}) \times (X/\mathcal{R}) \;-\; \Delta(X/\mathcal{R})$$

must be open in $(X/\mathcal{R}) \times (X/\mathcal{R})$. But this means that the diagonal $\Delta(X/\mathcal{R})$ must be a closed subset of $(X/\mathcal{R})\times(X/\mathcal{R})$ and therefore that $X/\mathcal{R}$ must satisfy the Hausdorff Separation Property.■

(c) The condition on $\Gamma_{\mathcal{R}}$ implies that each equivalence class is open. But this means that each point in $X/\mathcal{R}$ must be open and hence the latter must be discrete.■

9. (i) The binary relation $\mathcal{R}$ is symmetric and transitive but not symmetric. Therefore the equivalence relation $\mathcal{E}$ generated by $\mathcal{R}$ consists of the union $\mathcal{R}$ with the diagonal of D^2; in other words, $u\ \mathcal{E}\ v$ if and only if $u = v$ or $u\ \mathcal{R}\ v$, and if $u \neq v$ then $u\ \mathcal{E}\ v$ if and only if $u = \alpha v$, where $|u| = |v| = 1$ and $\alpha^d = 1$.■

(ii) In order to prove the existence of the continuous mapping h^* we need to show that $h(u) = h(v)$ if $u\ \mathcal{R}\ v$ and $u \neq v$; *i.e.*, $u = \alpha v$, where $|u| = |v| = 1$ and $\alpha^d = 1$. Under these conditions we have $h(z) = (0, z^d)$, so if u and v satisfy the given conditions then $h(u) = (0, u^d) = (0, \alpha^d v^d)$ because $\alpha^d = 1$. Therefore h is constant on equivalence classes, which implies the existence of h^*.

Since D^2 is compact and $\mathbb{C}^2$ is Hausdorff, by Theorem III.1.9, the mapping h^* is a homeomorphism onto its image if and only if it is 1–1. This is equivalent to showing that $h(u) = h(v)$ implies $u\ \mathcal{R}\ v$.

Suppose that $h(u) = h(v)$; taking coordinates, we must have $(1-|u|)u = (1-|v|)v$ and $u^d = v^d$. The latter implies that $|u|^d = |v|^d$, which in turn implies that $|u| = |v|$. There are now two cases. CASE 1: Suppose that $|u| = |v| < 1$. Then $|u| = |v|$ implies $1 - |u| = 1 - |v|$, and if we combine this with the equation for first coordinates we see that $u = v$. CASE 2: Suppose that $|u| = |v| = 1$. In this case $h(u) = (0, u^d)$ and $h(v) = (0, v^d)$, so if the two image points are equal then $1 = (u/v)^d$; therefore if $\alpha = u/v$, then $\alpha^d = 1$ and $u = \alpha\, v$, so that $u\ \mathcal{R}\ v$, which is what we wanted to show.■

(iii) Follow the hint. We know that $z \to z^d$ maps D^2 to itself, and if $z \neq 0$ then the equivalence class of z consists of all numbers of the form $\alpha\, z$, where $\alpha^d = 1$; in the exceptional case where $z = 0$, the equivalence class of z is merely $\{0\}$. In all cases we know that $u \sim v$ implies $u^d = v^d$, so if $h(z) = z^d$ then $u \sim v$ implies $h(u) = h(v)$, which means that $h = h^* \circ p$, where $p : D^2 \to D^2/\mathcal{F}$ is the quotient projection. The mapping h^* is onto because h is onto, so by Theorem III.1.9 it is only necessary to verify that h^* is 1–1, or equivalently that $h(u) = h(v)$ implies $u \sim v$. If $u \neq 0$, then $v \neq 0$ too and we have $u^p = v^p$; as in (ii), this implies that $u = \alpha\, v$ for some α such that $\alpha^d = 1$. On the other hand if $u = 0$ and $h(v) = h(0) = 0$, then $v = 0$ so that $u \sim v$ in this case too.■

V.2 : Sums and cutting and pasting

Additional exercises

1. ($\Longrightarrow$) If X is locally connected then so is every open subset. But each A_α is an open subset, so each is locally connected.■

($\Longleftarrow$) We need to show that for each $x \in X$ and each open set U containing x there is an open subset $V \subset U$ such that $x \in V$ and V is connected. There is a unique α such that $x = i_\alpha(a)$ for some $a \in A_\alpha$. Let $U_0 = i_\alpha^{-1}(U)$. Then by the local connectedness of A_α and the openness of U_0 there is an open connected set V_0 such that $x \in V_0 \subset U_0$. If $V = i_\alpha(V_0)$, then V has the required properties.■

2. X is compact if and only if each A_α is compact and there are only finitely many (nonempty) subsets in the collection.

The ($\Longrightarrow$) implication follows because each A_α is an open and closed subspace of the compact space X and hence compact, and the only way that the open covering $\{ A_\alpha \}$ of X, which consists of pairwise disjoint subsets, can have a finite subcovering is if it contains only finitely many subsets. To prove the reverse implication, one need only use a previous exercise which shows that a finite union of compact subspaces is compact.■

3. Since the exercise asks for details in a sketch to be filled in, we shall begin by reprinting this sketch: — *Let $A \subset S^2$ be the set of all points $(x, y, z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T[A] = A$ and $T[B] = B$ so that each of A and B (as well as their intersection) can be viewed as a union of equivalence classes for the equivalence relation that produces $\mathbb{RP}^2$. By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where sign$(z) = \pm 1$, and thus it follows that the image of B in the quotient space is homeomorphic to $B_+ \cong D^2$. Now consider A. There is a homeomorphism h from $S^1 \times [-1, 1]$ to A sending (x, y, t) to $(\alpha(t)x,\ \alpha(t)y,\ \frac{1}{2}t)$ where*

$$\alpha(t) \quad = \quad \sqrt{1 - \frac{t^2}{4}}$$

and by construction $h(-v) = -h(v)$. The image of A in the quotient space is thus the quotient of $S^1 \times [-1, 1]$ modulo the equivalence relation $u \sim v \Longleftrightarrow u = \pm v$. This quotient space is in turn homeomorphic to the quotient space of the upper semicircular arc S^1_+ (all points with nonnegative y-coordinate) modulo the equivalence relation generated by setting $(-1, 0, t)$ equivalent to $(1, 0, -t)$, which yields the Möbius strip. The intersection of this subset in the quotient with the image of B is just the image of the closed curve on the edge of B_+, which also represents the edge curve on the Möbius strip.

We shall go through the insertions needed at various steps in this argument.

Let $A \subset S^2$ be the set of all points $(x, y, z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T[A] = A$ and $T(B) = B$ [*etc.*]

This is true because if $T(v) = w$, then the third coordinates of both points have the same absolute values and of course they satisfy the same inequality relation with respect to $\frac{1}{2}$.

By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where sign$(z) = \pm 1$,

This is true the third coordinates of all points in B are nonzero.

There is a homeomorphism h from $S^1 \times [-1, 1]$ to A sending (x, y, t) to $(\alpha(t)x, \alpha(t)y, \frac{1}{2}t)$ where

$$\alpha(t)s \quad = \quad \sqrt{1 - \frac{t^2}{4}}$$

One needs to verify that h is 1–1 onto; this is essentially an exercise in algebra. Since we are dealing with compact Hausdorff spaces, continuous mappings that are 1–1 onto are automatically homeomorphisms.

This quotient space $[S^1 \times [-1, 1]$ *modulo the equivalence relation* $u \sim v \Longleftrightarrow u = \pm v]$ *is in turn homeomorphic to the quotient space of the upper semicircular arc* S^1_+ *(all points with nonnegative y-coordinate) modulo the equivalence relation generated by setting* $(-1, 0, t)$ *equivalent to* $(1, 0, -t)$, *which yields the Möbius strip.*

Let $\mathcal{A}$ and $\mathcal{B}$ be the respective equivalence relations on $S^1_+ \times [-1, 1]$ and $S^1 \times [-1, 1]$, and let $\mathbf{A}$ and $\mathbf{B}$ be the respective quotient spaces. By construction the inclusion $S^1_+ \times [-1, 1] \subset S^1 \times [-1, 1]$ passes to a continuous map of quotients, and it is necessary and sufficient to check that this map is 1–1 and onto. This is similar to a previous exercise. Points in $S^1 - S^1_+$ all have negative second coordinates and are equivalent to unique points with positive second coordinates. This implies that the mapping from $\mathbf{A}$ to $\mathbf{B}$ is 1–1 and onto at all points except perhaps those whose second coordinates are zero. For such points the equivalence relations given by $\mathcal{A}$ and $\mathcal{B}$ are identical, and therefore the mapping from $\mathbf{A}$ to $\mathbf{B}$ is also 1–1 and onto at all remaining points.■

4. We can and shall view X as $A \cup_{\text{id}} B$.

Consider the map $F_0 : A \sqcup B \to A \sqcup B$ defined by H^{-1} on A and the identity on B. We claim that this passes to a unique continuous map of quotients from X to $A \cup_h B$; *i.e.*, the map F_0 sends each nonatomic equivalence classes $\{ (c, 1), (c, 2) \}$ for $X = A \cup_{\text{id}} B$ to a nonatomic equivalence class of the form $\{ (u, 1), (h(u), 2) \}$ for $A \cup_h B$. Since F_0 sends $(c, 1)$ to $(h^{-1}(c), 1)$ and $(c, 2)$ to itself, we can verify the compatibility of F_0 with the equivalence relations by taking $u = h^{-1}(c)$. Passage to the quotients then yields the desired map $F : X \to A \cup_h B$.

To show this map is a homeomorphism, it suffices to define Specifically, start with $G_0 = F_0^{-1}$, so that $G_0 = H$ on A and the identity on B. In this case it is necessary to show that a nonatomic equivalence class of the form $\{ (u, 1), (h(u), 2) \}$ for $A \cup_h B$ gets sent to a nonatomic equivalence class of the form $\{ (c, 1), (c, 2) \}$ for $X = A \cup_{\text{id}} B$. Since G_0 maps the first set to $\{ (h(u), 1), (h(u), 2) \}$ this is indeed the case, and therefore G_0 also passes to a map of quotients which we shall call G.

Finally we need to verify that F and G are inverses to each other. By construction the maps F_0 and G_0 satisfy $F([y]) = [F_0(y)]$ and $G([z]) = [G_0(z)]$, where square brackets denote equivalence classes. Therefore we have

$$G \circ F([y]) = G\left([F_0(y)]\right) = [G_0\left(F_0(y)\right)]$$

which is equal to $[y]$ because F_0 and G_0 are inverse to each other. Therefore $G \circ F$ is the identity on X. A similar argument shows that $F \circ G$ is the identity on $A \cup_h B$.

To construct the example where X is **not** homeomorphic to $A \cup_h B$, we follow the hint and try to find a homeomorphism of the four point space $\{\pm 1\} \times \{1, 2\}$ to itself such that X is **not** homeomorphic to $A \cup_h B$ is connected; this suffices because we know that X is not connected. Sketches on paper or physical experimentation with wires or string are helpful in finding the right formula.

Specifically, the homeomorphism we want is given as follows:

$$\begin{array}{rcl} (-1,1) \in A_+ & \longrightarrow & (1,2) \in A_- \\ (1,1) \in A_+ & \longrightarrow & (1,1) \in A_- \\ (1,2) \in A_+ & \longrightarrow & (-1,1) \in A_- \\ (-1,2) \in A_+ & \longrightarrow & (-1,2) \in A_- \end{array}$$

The first of these implies that the images of $S^1_+ \times \{2\}$ and $S^1_- \times \{1\}$ lie in the same component of the quotient space, the second of these implies that the images of $S^1_- \times \{1\}$ and $S^1_+ \times \{1\}$ both lie in the same component, and the third of these implies that the images of $S^1_+ \times \{2\}$ and $S^1_- \times \{2\}$ also lie in the same component. Since the entire space is the union of the images of the connected subsets $S^1_\pm \times \{1\}$ and $S^1_\pm \times \{2\}$ it follows that $A \cup_h B$ is connected.■

FOOTNOTE. The argument in the first part of the exercise remains valid if A and B are open rather than closed subsets.■

5. (a) For each j let $\mathbf{in}_j : X_j \to \coprod_k X_k$ be the standard injection into the disjoint union, and let

$$P : \coprod_k X_k \longrightarrow \bigvee_k (X_k, x_k)$$

be the quotient map defining the wedge. Define Y_j to be $P \circ \mathbf{in}_j[\,X_j\,]$. By construction the map $P \circ \mathbf{in}_j$ is continuous and 1–1; we claim it also sends closed subsets of X_j to closed subsets of the wedge. Suppose that $F \subset X_j$ is closed; then $P \circ \mathbf{in}_j[F]$ is closed in the wedge if and only if its inverse image under P is closed. But this inverse image is the union of the closed subsets $\mathbf{in}_j[F]$ and $\coprod_k \{x_k\}$ (which is a finite union of one point subsets that are assumed to be closed). It follows that Y_j is homeomorphic to $X + j$. The condition on $Y_k \cap Y_\ell$ for $k \neq \ell$ is an immediate consequence of the construction.

The assertion that the wedge is Hausdorff if and only if each summand is follows because a subspace of a Hausdorff space is Hausdorff, and a finite union of closed Hausdorff subspaces is always Hausdorff (by a previous exercise).

To verify the assertions about compactness, note first that for each j there is a continuous collapsing map q_j from $\vee_k\ (X_k, x_k)$ to (X_j, x_j), defined by the identity on the image of (X_j, x_j) and by sending everything to the base point on every other summand. If the whole wedge is compact, then its continuous under q_j, which is the image of X_j, must also be compact. Conversely if the sets X_j are compact for all j, then the (finite!) union of their images, which is the entire wedge, must be compact.

To verify the assertions about connectedness, note first that for each j there is a continuous collapsing map q_j from $\vee_k\ (X_k, x_k)$ to (X_j, x_j), defined by the identity on the image of (X_j, x_j) and by sending everything to the base point on every other summand. If the whole wedge is connected, then its continuous under q_j, which is the image of X_j, must also be connected. Conversely if the sets X_j are connected for all j, then the union of their images, which is the entire wedge, must be connected because all these images contain the base point. Similar statements hold for arcwise connectedness and follow by inserting "arcwise" in front of "connected" at every step of the argument.■

(b) To prove existence, first observe that there is a unique continuous map $\widetilde{F} : \coprod_k X_k \to Y$ such that $\mathbf{in}_j \circ \widetilde{F} = F_j$ for all j. This passes to a unique continuous map F on the quotient space $\vee_k\ (X_k, x_k)$ because $\widetilde{F}$ is constant on the equivalence classes associated to the quotient projection P. This constructs the map we want; uniqueness follows because the conditions prescribe the definition at every point of the wedge.■

(c) Strictly speaking, one should verify that the so-called weak topology is indeed a topology on the wedge. We shall leave this to the reader.

To prove [1], note that ($\Longrightarrow$) is trivial. For the reverse direction, we need to show that if E is closed in Y then $h^{-1}[E]$ is closed with respect to the so-called weak topology we have defined. The subset in question is closed with respect to this topology if and only if $h^{-1}[E] \cap \varphi[X_j]$ is closed in $\varphi[X_j]$ for all j, and since φ_j maps its domain homeomorphically onto its image, the latter is true if and only if $\varphi^{-1} \circ h^{-1}[E]$ is closed in X_j for all j. But these conditions hold because each of the maps $\varphi_j \circ h$ is continuous. To prove [2], note first that there is a unique set-theoretic map, and then use [1] to conclude that it is continuous.■

(d) For each j let $y_j \in X_j$ be a point other than x_j, and consider the set E of all points y_j. This is a closed subset of the wedge because its intersection with each set $\varphi[X_j]$ is a one point subset and hence closed. In fact, every subset of E is also closed by a similar argument (the intersections with the summands are either empty or contain only one point), so E is a discrete closed subset of the wedge. Compact spaces do not have infinite discrete closed subspaces, and therefore it follows that the infinite wedge with the weak topology is not compact.■

We shall conclude this document by filling in some details in the final remark in the exercises for Section V.2. This remark is reprinted here for the sake of convenience:

Remark. If each of the summands in (d) is compact Hausdorff, then there is a natural candidate for a *strong topology* on a countably infinite wedge which makes the latter into a compact Hausdorff space. In some cases this topology can be viewed more geometrically; for example, if each (X_j, x_j) is equal to $(S^1, 1)$ and there are countably infinitely many of them, then the space one obtains is the *Hawaiian earring* in $\mathbb{R}^2$ given by the union of the circles defined by the equations

$$\left(x - \frac{1}{2^k}\right)^2 \ + \ + \ y^2 \quad = \quad \frac{1}{2^{2k}} \ .$$

As usual, drawing a picture may be helpful. The k^{th} circle has center $(1/2^k,\, 0)$ and passes through the origin; the y-axis is the tangent line to each circle at the origin.

SKETCHES OF VERIFICATIONS OF ASSERTIONS IN THIS REMARK.

If we are given an infinite sequence of compact Hausdorff pointed spaces $\{\,(X_n, x_n)\,\}$ we can put a compact Hausdorff topology on their wedge as follows. Let W_k be the wedge of the first k spaces; then for each k there is a continuous map

$$q_k : \bigvee_n (X_n, x_n) \quad \longrightarrow \quad W_k$$

(with the so-called weak topology on the wedge) that is the identity on the first k summands and collapses the remaining ones to the base point. These maps are in turn define a continuous function

$$\mathbf{q} : \bigvee_n (X_n, x_n) \quad \longrightarrow \quad \prod_k W_k$$

whose projection onto W_k is q_k. This mapping is continuous and 1–1; if its image is closed in the (compact!) product topology, then this defines a compact Hausdorff topology on the infinite wedge $\vee_n(X_n, x_n)$.

Here is one way of verifying that the image is closed. For each k let $c_k : W_k \to W_{k-1}$ be the map that is the identity on the first $(k-1)$ summands and collapses the last one to a point. Then we may define a continuous map C on $\prod_{k\geq 1} W_k$ by first projecting onto the product $\prod_{k\geq 2} W_k$ (forget the first factor) and then forming the map $\prod_{k\geq 2} W_k$. The image of $\mathbf{q}$ turns out to be the set of all points $\mathbf{x}$ in the product such that $C(\mathbf{x}) = \mathbf{x}$. Since the product is Hausdorff the image set is closed in the product and thus compact.

A comment about the compactness of the Hawaiian earring E might be useful. Let F_k be the union of the circles of radius 2^{-j} that are contained in E, where $j \leq k$, together with the closed disk bounded by the circle of radius $2^{-(k+1)}$ in E. Then F_k is certainly closed and compact. Since E is the intersection of all the sets F_k it follows that E is also closed and compact.■