SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 5

Fall 2014

VI. Spaces with additional properties

VI.1 : Second countable spaces

Problems from Munkres, § 30, *pp.* 194 – 195

9. [*First part only*] The statement and proof are parallel to a result about compact spaces in the course notes, the only change being that "compact" is replaced by "Lindelöf."■

10. Suppose first that X is a finite product of spaces Y_i such that each Y_i has a countable dense subset D_i. Then $\prod_i D_i$ is countable and

$$X \;=\; \prod_i Y_i \;=\; \prod_i \overline{D_i} \;=\; \overline{\prod_\alpha D_i}\;.$$

Suppose now that X is countably infinite. The same formula holds, but the product of the D_i's is not necessarily countable. To adjust for this, pick some point $\delta_j \in D_j$ for each j and consider the set E of all points $(a_0, a_1, \cdots)$ in $\prod_j D_j$ such that $a_j = \delta_j$ for all but at most finitely many values of j. This set is countable, and we claim it is dense. It suffices to show that every basic open subset contains at least one point of E. But suppose we are given such a set $V = \prod_j V_j$ where V_j is open in X_j and $V_j = X_j$ for all but finitely many j; for the sake of definiteness, suppose this happens for $j > M$. For $j \leq M$, let $b_j \in D_j \cap V_j$; such a point can be found since D_j is dense in X_j. Set $b_j = \delta_j$ for $j > M$. If we let $b = (b_0, b_1, \cdots)$, then it then follows that $b \in E \cap V$, and this implies E is dense in the product.■

13. This is similar to the proof that an open subset of $\mathbb{R}^n$ has only countably many components.■

14. This is essentially the same argument as the one showing that a product of two compact spaces is compact. The only difference is that after one constructs an open covering of Y at one step in the proof, then one only has a countable subcovering and this leads to the existence of a countable subcovering of the product. Details are left to the reader.■

Additional exercises

1. List the basic open subsets in $\mathcal{B}$ as a sequence $U_0, U_1, U_2, \cdots$ If W is an open subset of U define a function $\psi_W : \mathbb{N} \to \{0,1\}$ by $\psi_W(i) = 1$ if $U_i \subset W$ and 0 otherwise. Since $\mathcal{B}$ is a basis, every open set W is the union of the sets U_i for which $\psi_W(i) = 1$. In particular, the latter implies that if $\psi_V = \psi_W$ then $V = W$ and hence we have a 1–1 map from the set of all open subsets to the set of functions from $\mathbb{N}$ to $\{0,1\}$. Therefore the cardinality of the family of open subsets is at most the cardinality of the set of functions, which is $2^{\aleph_0}$.

If X is Hausdorff, or even if we only know that every one point subset of X is closed in X, then we may associate to each $x \in X$ the open subset $X - \{x\}$. If $x \neq y$ then $X - \{x\} \neq X - \{y\}$

and therefore the map $C : X \to \mathbf{T}$ defined in this fashion is 1–1. Therefore we have $|X| \leq |\mathbf{T}|$, and therefore by the preceding paragraph we know that $|X| \leq 2^{\aleph_0}$.■

A somewhat more complicated argument yields similar conclusions for spaces satisfying the weaker $\mathbf{T_0}$ condition stated in Section VI.3.■

2. (*a*) The x-axis is closed because it is closed in the ordinary Euclidean topology and the "new" topology contains the Euclidean topology; therefore the x-axis is closed in the "new" topology. The subspace topology on the x-axis is the discrete topology intersection of the open set $T_\varepsilon(x)$ with the real axis is $\{ x \}$.■

(*b*) In general if $(X, \mathbf{T})$ is Hausdorff and $\mathbf{T} \subset \mathbf{T}^*$ then $(X, \mathbf{T}^*)$ is also Hausdorff, for a pair of disjoint $\mathbf{T}$-open subsets containing distinct point u and v will also be a pair of $\mathbf{T}$-open subsets with the same properties.■

(*c*) To show that a subset D of a topological space X is dense, it suffices to show that the intersection of D with every nonempty open subset in some base $\mathcal{B}$ is nonempty (why?). Thus we need to show that the generating set is a base for the topology and that every basic open subset contains a point with rational coordinates.

The crucial point to showing that we have a base for the topology is to check that if U and V are basic open subsets containing some point $z = (x, y)$ then there is some open subset in the generating collection that contains z and is contained in $U \cap V$. If U and V are both basic open subsets we already know this, while if $z \in T_\varepsilon(a) \cap T_\delta(b)$ there are two cases depending upon whether or not z lies on the x-axis; denote the latter by A. If $z \notin A$ then it lies in the metrically open subsets $T_\varepsilon(a) - A$ and $T_\varepsilon(b) - A$, and one can find a metrically open subset that contains z and is contained in the intersection. On the other hand, if $z \in A$, then $T_\varepsilon(a) \cap A = \{a\}$ and $T_\delta(b) \cap A = \{b\}$ imply $a = b$, and the condition on intersections is immediate because the intersection of the subsets is either $T_\varepsilon(a)$ or $T_\delta(a)$ depending upon whether $\delta \leq \varepsilon$ or vice versa.■

3. For each property $\mathcal{P}$ given in the exercise, the space X has property $\mathcal{P}$ if and only if each A_α does and there are only finitely many α for which A_α is nonempty. The verifications for the separate cases are different and will be given in reverse sequence.

The Lindelöf property.

The proof in this case is the same as the proof we gave for compactness in an earlier exercises with "countable" replacing "finite" throughout.■

Separability.

($\Longrightarrow$) Let D be the countable dense subset. Each A_α must contain some point of D, and by construction this point is not contained in any of the remaining sets A_β. Thus we have a 1–1 function from $\mathcal{A}$ to D sending α to a point $d(\alpha) \in A_\alpha \cap D$. This implies that the cardinality of $\mathcal{A}$ is at most $|D| \leq \aleph_0$.■

($\Longleftarrow$) If D_α is a dense subset of A_α and $\mathcal{A}$ is countable, then $\cup_\alpha D_\alpha$ is a countable dense subset of $\mathcal{A}$.■

Second countability.

($\Longrightarrow$) Since a subspace of a second countable space is second countable, each A_α must be second countable. Since the latter condition implies both separability and the Lindelöf property, the preceding arguments show that only countably many summands can be nontrivial.

($\Longleftarrow$) If $\mathcal{A}$ is countable and $\mathcal{B}_\alpha$ is a countable base for A_α then $\cup_\alpha \mathcal{B}_\alpha$ determines a countable base for X (work out the details!).■

4. More generally, if $\mathcal{B}_X$ is a base for the topology on X and $\mathcal{B}_Y$ is a base for the topology on Y, then a base for the topology on $X \times Y$ is given by all sets of the form $U \times V$ where $U \in \mathcal{B}_X$ and

$V \in \mathcal{B}_Y$. — To prove this, it is enough to show it for basic open subsets $M \times N$ in $X \times Y$. We then have $M = \cup_i U_i$ and $N = \cup_j V_j$ for $U_i \in \mathcal{B}_X$ and $V_j \in \mathcal{B}_Y$, which means that $M \times N = \cup_{i,j} U_i \times V_j$.

If both bases are countable, then the unions in the preceding sentence can be taken to be countable if we remove duplicate summands (there are only countably many possibilities for U_i or V_j), and this implies that the indexing set for the decomposition $M \times N = \cup_{i,j} U_i \times V_j$ is also countable.■

VI.2 : Compact spaces – II

Problems from Munkres, § 28, pp. 181 – 182

6. Follow the hint to define a and the sequence. The existence of ε is guaranteed because $a \notin f[X]$ and the compactness of $f[X]$ imply that the continuous function $g(x) = \mathbf{d}(a, f(x))$ is positive valued, and it is bounded away from 0 because it attains a minimum value. Since $x_k \in f[X]$ for all k it follows that $\mathbf{d}(a, x_k) > \varepsilon$ for all k. Given $n \neq m$ write $m = n + k$; reversing the roles of m and n if necessary we can assume that $k > 0$. If f is distance preserving, then so is every n-fold iterated composite $\circ^n f$ of f with itself. Therefore we have that

$$\mathbf{d}(x_n, x_m) \quad = \quad \mathbf{d}\left(\circ^n f(a), \circ^n f(x_k)\right) \quad = \quad \mathbf{d}(a, x_k) \quad > \quad \varepsilon$$

for all distinct nonnegative integers m and n. But this cannot happen if X is compact, because the latter implies that $\{ x_n \}$ has a convergent subsequence. This contradiction implies that our original assumption about the existence of a point $a \notin f[X]$ is false, so if is onto. On the other hand $x \neq y$ implies

$$0 \quad < \quad \mathbf{d}(x, y) \quad = \quad \mathbf{d}\left(f(x), f(y) \right)$$

and thus that f is also 1–1. Previous results now also imply that f is a homeomorphism onto its image.■

FOOTNOTE. To see the need for compactness rather than (say) completeness, consider the map $f(x) = x + 1$ on the set $[0, +\infty)$ of nonnegative real numbers.■

Additional exercises

1. Follow the steps in the hint.

> *A map g is $1-1$ on a subset B is and only if $B \times B \cap (g \times g)^{-1}[\,\Delta_Y\,] = \Delta_B$ where Δ_S denotes the diagonal in $S \times S$.*

This is true because the set on the left hand side of the set-theoretic equation is the set of all (b, b') such that $g(b) = g(b')$. If there is a nondiagonal point in this set then the function is not 1–1, and conversely if the function is 1–1 then there cannot be any off-diagonal terms in the set.■

> *Show that if $D' = (f \times f)^{-1}[\,\Delta_Y\,] - \Delta_X$, then D' is closed and disjoint from the diagonal.*

Since f is locally 1–1, for each $x \in X$ there is an open set U_x such that $f|U_x$ is 1–1; by the first step we have

$$U_x \times U_x \cap (f \times f)^{-1}[\,\Delta_Y\,] \quad = \quad \Delta_{U_x} \ .$$

If we set $W = \cap_x \, U_x \times U_x$ then W is an open neighborhood of Δ_X and by construction we have

$$(f \times f)^{-1}[\,\Delta_Y\,] \;\cap\; W \quad = \quad \Delta_X$$

which shows that Δ_X is open in $(f \times f)^{-1}[\,\Delta_Y\,]$ and thus its relative complement in the latter — which is D' — must be closed.■

Show that the subsets D' and $A \times A$ are disjoint, and find a square neighborhood of $A \times A$ disjoint from D'.

Since $f|A$ is 1–1 we have $A \times A \cap (f \times f)^{-1}[\,\Delta_Y\,] = \Delta_A$, which lies in $\Delta_X = (f \times f)^{-1}[\,\Delta_Y\,] - D'$. Since A is a compact subset of the open set $X \times X - D'$, by Wallace's Theorem there is an open set U such that

$$A \times A \quad \subset \quad U \times U \quad \subset \quad X \times X - D' \ .$$

The first part of the proof now implies that $f|U$ is 1–1.■

2. By the Inverse Function Theorem we know that f is locally 1–1, and therefore by the previous exercise we know that f is 1–1 on an open set V such that $A \subset V \subset U$. But the Inverse Function Theorem also implies that f locally has a $\mathbf{C}^1$ inverse on U. Since f has a global set-theoretic inverse from $f[V]$ back to V, it follows that this global inverse is also $\mathbf{C}^1$.■

3. As usual, we follow the steps in the hint.

For each integer $r > 0$ show that every integer is within p^{-r} of one of the first p^{r+1} nonnegative integers.

If n is an integer, use the long division property to write $n = p^{r+1}a$ where $0 \leq a < p^{r+1}$. We then have $\mathbf{d}_p(n, a) \leq p^{-(r+1)} < p^r$.■

Furthermore, each open neighborhood of radius p^{-r} centered at one of these integers a is a union of p neighborhoods of radius $p^{-(r+1)}$ over all of the first p^{r+2} integers b such that $b \equiv a \bmod p^{r+1}$.

Suppose we do long division by p^{r+2} rather than p^{r+1} and get a new remainder a'. How is it related to a? Very simply, a' must be one of the p nonnegative integers b such that $0 \leq b < p^{r+2}$ and $b \equiv a \bmod p^{r+1}$. Since $\mathbf{d}(u, v) < p^{-r}$ if and only if $\mathbf{d}_p(u, v,) \leq p^{-(r+1)}$, it follows that $N_{p^{-r}}(a)$ is a union of p open subsets of the form $N_{p^{-(r+1)}}(b)$ as claimed (with the numbers b satisfying the asserted conditions).

Find a sequence of nonnegative integers $\{\, b_r \,\}$ such that the open neighborhood of radius p^{-r} centered at b_r contains infinitely many terms in the sequence and $b_{r+1} \equiv b_r \bmod p^{r+1}$.

This is very similar to a standard proof of the Bolzano-Weierstrass Theorem in real variables. Infinitely many terms of the original sequence must lie in one of the sets $N_1(b_0)$ where $1 \leq b < p$. Using the second step we know there is some b_1 such that $b_1 \equiv b_0 \bmod p$ and infinitely many terms of the sequence lie in $N_{p^{-1}}(b_1)$, and one can continue by induction (fill in the details!).■

Form a subsequence of $\{\, a_n \,\}$ by choosing distinct points $a_{n(k)}$ recursively such that $n(k) > n(k-1)$ and $a_{n(k)} \in N_{p^{-k}}(b_k)$. Prove that this subsequence is a Cauchy sequence and hence converges.]

We know that the neighborhoods in question contain infinitely many values of the sequence, and this allows us to find $n(k)$ recursively. It remains to show that the construction yields a Cauchy sequence. The key to this is to observe that $a_{n(k)} \equiv b_k \bmod p^{k+1}$ and thus we also have

$$\mathbf{d}_p \left(a_{n(k+1)}, a_{n(k)} \right) \quad < \quad \frac{1}{p^k} \ .$$

Similarly, if $\ell > k$ then we have

$$\mathbf{d}_p\left(a_{n(\ell)}, a_{n(k)}\right) \quad < \quad \frac{1}{p^{\ell-1}} \; + \; \cdots \; + \; \frac{1}{p^k} \; .$$

Since the geometric series $\sum_k p^{-k}$ converges, for every $\varepsilon > 0$ there is an M such that $\ell, k \geq M$ implies the right had side of the displayed inequality is less than ε, and therefore it follows that the constructed subsequence is indeed a Cauchy sequence. By completeness (of the completion) this sequence converges. Therefore $\widehat{\mathbb{Z}_p}$ is (sequentially) compact.■

VI.3 : Separation axioms

Problem from Munkres, § 26, *pp.* 170 − 172

11. Follow the suggestion of the hint to define C, D and to find U, V. There are disjoint open subset sets U, $V \subset X$ containing C and D respectively because a compact Hausdorff space is $\mathbf{T_4}$.

For each α the set $B_\alpha = A_\alpha - (U \cup V)$ is a closed and hence compact subset of X. If each of these subsets is nonempty, then the linear ordering condition implies that the family of closed compact subsets B_α has the finite intersection property; specifically, the intersection

$$B_{\alpha(1)} \cap \ \ldots \ \cap B_{\alpha(k)}$$

is equal to $B_{\alpha(j)}$ where $A_{\alpha(j)}$ is the smallest subset in the linearly ordered collection

$$\{ \, A_{\alpha(1)}, \ \ldots \ , A_{\alpha(k)} \, \} \ .$$

Therefore by compactness it will follow that the intersection

$$\bigcap_\alpha B_\alpha \quad = \quad \left(\bigcap_\alpha A_\alpha \right) - (U \cup V)$$

is nonempty. But this contradicts the conditions $\cap_\alpha \, A_\alpha \quad = \quad C \cup D \quad \subset \quad U \cup V$. Therefore it follows that the $\cap_\alpha A_\alpha$ must be connected.

Therefore we only need to answer the following question: Why should the sets $A_\alpha - (U \cup V)$ be nonempty? If the intersection is empty then $A_\alpha \subset U \cup V$. By construction we have $C \subset A_\alpha \cap U$ and $D \subset A_\alpha \cap V$, and therefore we can write A_α as a union of two nonempty disjoint open subsets. However, this contradicts our assumption that A_α is connected and therefore we must have $A_\alpha - (U \cup V) \neq \emptyset$.■

Problems from Munkres, § 33, *pp.* 212 − 214

2. (*a*) [*For metric spaces.*]

The proof is based upon Urysohn's Lemma and therefore is valid in arbitrary $\mathbf{T_4}$ spaces; we have stated it only for metric spaces because we have only established Urysohn's Lemma in that case.

Let X be the ambient topological space and suppose that u and v are distinct points of X. Then $\{u\}$ and $\{v\}$ are disjoint closed subsets and therefore there is a continuous function $f : X \to \mathbb{R}$

such that $f(u) = 0$ and $f(v) = 1$. Since $f[X]$ is connected, for each $t \in [0,1]$ there is a point $x_t \in X$ such that $f(x_t) = t$. Since $s \neq t$ implies $f(x_s) \neq f(x_t)$, it follows that the map $x : [0,1] \to X$ sending t to x_t is 1–1. Therefore we have $2^{\aleph_0} \leq |X|$, and hence X is uncountable.∎

6. (*a*) Let $(X, \mathbf{d})$ be a metric space, let $A \subset X$ be closed and let $f(x) = \mathbf{d}(x, A)$. Then

$$A \quad = \quad f^{-1}(\{0\}) \quad = \quad \bigcap_n f^{-1}\Big[\,\big(-\tfrac{1}{n}, \tfrac{1}{n}\big)\,\Big]$$

presents A as a countable intersection of open subsets.∎

8. For each $a \in A$ there is a continuous function $f_a : X \to [-1, 1]$ that is -1 at a and 0 on B. Let $U_a = f_a^{-1}[[-1, 0)]$. Then the sets U_a define an open covering of A and hence there is a finite subcovering corresponding to $U_{a(1)}, \cdots, U_{a(k)}$. Let f_i be the function associated to $a(i)$, let g_i be the maximum of f_i and 0 (so g_i is continuous by a previous exercise), and define

$$f \quad = \quad \prod_{i=1}^{k} g_k \ .$$

By construction the value of f is 1 on B because each factor is 1 on B, and $f = 0$ on $\cup_i U_{a(i)}$ because $g_j = 0$ on $U_{a(j)}$; since the union contains A, it follows that $f = 0$ on A.∎

Additional exercises

1. The strict containment condition implies that the identity map from $(X, \mathbf{T})$ to $(X, \mathbf{T}^*)$ is continuous but not a homeomorphism. Since the image of a compact set is compact, it follows that $(X, \mathbf{T}^*)$ is compact. If it were Hausdorff, the identity map would be closed and thus a homeomorphism. Therefore $(X, \mathbf{T}^*)$ is not Hausdorff.∎

2. (*a*) ($\Longrightarrow$) Suppose that X is $\mathbf{T_3}$, let $x \in X$ and let V be a basic open subset containing x. Then there is an open set U in X such that $x \in U \subset \overline{U} \subset V$. Since $\mathcal{B}$ is a basis for the topology, one can find a basic open subset W in $\mathcal{B}$ such that $x \in W \subset U$, and thus we have $x \in W \subset \overline{W} \subset \overline{U} \subset V$.∎

($\Longleftarrow$) Suppose that X is $\mathbf{T_1}$ and satisfies the condition in the exercise. If U is an open subset and $x \in U$, let V be a basic open set from $\mathcal{B}$ such that $x \in V \subset U$, and let W be the basic open set that exists by the hypothesis in the exercise. Then we have $x \in W \subset \overline{W} \subset V \subset U$ and therefore X is regular.∎

(*b*) By the first part of the exercise we only need to show this for points in basic open subsets. If the basic open subset comes from the metric topology, this follows because the metric topology is $\mathbf{T_3}$; note that the closure in the new topology might be smaller than the closure in the metric topology, but if a metrically open set contains the metric closure it also contains the "new" closure. If the basic open subset has the form $T_\varepsilon(a)$ for some a and z belongs to this set, there are two cases depending upon whether or not z lies on the x-axis, which we again call A. If $z \notin A$, then z lies in the metrically open subset $T_\varepsilon(a) - A$, and one gets a subneighborhood whose closure lies in the latter exactly as before. On the other hand, if $z \in A$ then $z = (a, 0)$ and the closure of the set $T_\delta(a)$ in either topology is contained in the set $T_\varepsilon(a)$.∎

3. (*a*) If X is $\mathbf{T_3}$, then for each point $x \notin A$ there are disjoint open subsets U and V such that $x \in U$ and $A \subset V$. Let $\pi : X \to X/A$ be the quotient projection; we claim that $\pi[U]$ and $\pi[V]$ are open disjoint subsets of X/A. Disjointness follows immediately from the definition of the equivalence relation, and the sets are open because their inverse images are the open sets $U = \pi^{-1}\,[\,\pi[U]\,]$ and $V = \pi^{-1}\,[\,\pi[V]\,)$ respectively.■

(b) Suppose that $F \subset X$ is closed; we need to show that $\pi^{-1}\,[\,\pi(F)\,]$ is also closed. There are two cases depending upon whether or not $A \cap F = \emptyset$. If the two sets are disjoint, then $\pi^{-1}\,[\,\pi(F)\,] = F$, and if the intersection is nonempty then $\pi^{-1}\,[\,\pi(F)\,] = F \cup A$. In either case the inverse image is closed, and therefore the image of F is always closed in the quotient space.

We claim that π is not necessarily open if A has a nonempty interior. Suppose that both of the statements in the previous sentence are true for a specific example, and let v be a nonempty open subset of X that is contained in A. If π is open then $\pi[V] = \pi[A] = \{A\} \in X/A$ is an open set. Therefore its inverse image, which is A, must be open in X. But it is also closed in X. Therefore we have the following conclusion: *If A is a nonempty proper closed subset of the connected space X with a nonempty interior, then $\pi : X \to X/A$ is not an open mapping.*■

(c) Let $p : X \to X/A$ and $q : Y \to Y/B$ be the projection maps, and consider the composite $q\circ f$. Then the condition $f[A] \subset B$ implies that $q \circ f$ sends each equivalence class for the relation defining X/A to a point in Y/B, and thus by the basic properties of quotient maps it follows that there is a unique continuous map $F : X/A \to Y/B$ such that $q \circ f = F \circ p$; this is equivalent to the conclusion stated in this part of the problem.■

4. The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x$ defines a homeomorphism from $(\mathbb{R}, \mathbf{U}) \to (\mathbb{R}, \mathbf{L})$, so it suffices to consider the case of the lower semicontinuity topology $\mathbf{U}$.

We shall first prove that $(\mathbb{R}, \mathbf{U})$ is $\mathbf{T}_0$. Given two points in $\mathbb{R}$ we can label them as b_1, b_2 such that $b_1 < b_2$. If $a = \frac{1}{2}(b_a + b_2)$, then (a, ∞) is an open subset in $\mathbf{U}$ which contains b_2 but not b_1. — In contrast, note that there is no open subset which contains b_1 but not b_2.■

To see that $(\mathbb{R}, \mathbf{U})$ is not $\mathbf{T}_1$, we need to show that for each $x \in \mathbb{R}$ the set $\mathbb{R} - \{x\}$ is not in $\mathbf{U}$. By definition, every subset in $\mathbf{U}$ is connected in the metric topology, and since $\mathbb{R} - \{x\}$ is not connected in the metric topology it follows that $\mathbb{R} - \{x\}$ is not in $\mathbf{U}$.■

VI.4 : Local compactness and compactifications

Problems from Munkres, § 38, pp. 241 – 242

2. We need to begin by describing the compactification mentioned in the exercise. It is given by taking the closure of the embedding (homeomorphism onto its image) $g : (0, 1) \to [-1, 1]^2$ that is inclusion on the first coordinate and $\sin(1/x)$ on the second.

Why is it impossible to extend $\cos(1/x)$ to the closure of the image? Look at the points in the image with coordinates $(1/k\pi, \sin k\pi)$ where k is a positive integer. The second coordinates of these points are always 0, so this sequence converges to the origin. If there is a continuous extension F, it will follow that

$$F(0,0) \quad = \quad \lim_{k\to\infty} \cos\left(\frac{1}{1/k\pi}\right) \quad = \quad \cos k\pi \ .$$

But the terms on the right hand side are equal to $(-1)^k$ and therefore do not have a limit as $k \to \infty$. Therefore no continuous extension to the compactification exists.

One can construct a compactification on which x, $\sin(1/x)$ and $\cos(1/x)$ extend by taking the closure of the image of the embedding $h : (0, 1) \to [-1, 1]^3$ defined by

$$h(x) \quad = \quad (x, \sin(1/x), \cos(1/x)\,) \ .$$

The continuous extensions are given by restricting the projections onto the first, second and third coordinates.■

3. [*Just give a necessary condition on the topology of the space.*]

If A is a dense subset of a compact metric space, then A must be second countable because a compact metric space is second countable and a subspace of a second countable space is also second countable.

This condition is also sufficient, but the sufficiency part was not assigned because it requires the Urysohn Metrization Theorem. The latter says that a $\mathbf{T_3}$ and second countable topological space is homeomorphic to a product of a countably infinite product of copies of $[0,1]$; this space is compact by Tychonoff's Theorem, and another basic result not covered in the course states that a countable product of metrizable spaces is metrizable in the product topology (see Munkres, Exercise 3 on pages 133–134). So if X is metrizable and second countable, the Urysohn Theorem maps it homeomorphically to a subspace of a compact metrizable space, and the closure of its image will be a metrizable compactification of X.■

Additional exercises

Definition. If $f : X \to Y$ is continuous, then f is *proper* (or *perfect*) if for each compact subset $K \subset Y$ the inverse image $f^{-1}[K]$ is a compact subset of X.

1. ($\Longrightarrow$) Suppose that f is proper and U is open in $Y^\bullet$. There are two cases depending upon whether $\infty_Y \in U$. If not, then $U \subset Y$ and thus $[f^\bullet]^{-1}[U] = f^{-1}[U]$ is an open subset of X; since X is open in $X^\bullet$ it follows that $f^{-1}[U]$ is open in $X^\bullet$. On the other hand, if $\infty_Y \in U$ then $Y - U$ is compact, and since f is proper it follows that

$$C \quad = \quad f^{-1}[Y - U] \quad = \quad X - f^{-1}\left[U - \{\infty_Y\}\right]$$

is a compact, hence closed, subset of X and $X^\bullet$. Therefore

$$[f^\bullet]^{-1}[U] \quad = \quad f^{-1}[U] \cup \{\infty_X\} \quad = \quad X^\bullet - C$$

is an open subset of $X^\bullet$.■

($\Longleftarrow$) Suppose that $f^\bullet$ is continuous and $A \subset Y$ is compact. Then

$$f^{-1}[A] = (f^\bullet)^{-1}[A]$$

is a closed, hence compact subset of $X^\bullet$ and likewise it is a compact subset of X.■

2. Let $f : X \to Y$ be a proper map of noncompact locally compact Hausdorff spaces, and let $f^\bullet$ be its continuous extension to a map of one point compactifications. Since the latter are compact Hausdorff it follows that $f^\bullet$ is closed. Suppose now that $F \subset X$ is closed. If F is compact, then so is $f[F]$ and hence the latter is closed in Y. Suppose now that F is not compact, and consider the closure E of F in $X^\bullet$. This set is either F itself or $F \cup \{\infty_X\}$ (since F is its own closure in X it follows that $E \cap X = F$). Since the closed subset $E \subset X^\bullet$ is compact, clearly $E \neq F$, so this implies the second alternative. Once again we can use the fact that $f^\bullet$ is closed to show that $f^\bullet[E] = f[F] \cup \{\infty_Y\}$ is closed in $Y^\bullet$. But the latter equation implies that $f[F] = f^\bullet[F] \cap Y$ is closed in Y.■

3. Write the polynomial as

$$p(z) \quad = \quad \sum_{k=0}^{n} a_k\, z^k$$

where $a_n \neq 0$ and $n > 0$, and rewrite it in the following form:

$$a_n z^n \cdot \left(1 \quad + \quad \sum_{k=1}^{n-1} \frac{a_k}{a_n} \frac{1}{z^{n-k}} \right)$$

The expression inside the parentheses goes to 1 as $n \to \infty$, so we can find $N_0 > 0$ such that $|z| \geq N_0$ implies that the absolute value (or modulus) of this expression is at least $\frac{1}{2}$.

Let $M > 0$ be arbitrary, and define

$$N_1 \quad = \quad \left(\frac{2M_0}{|a_n|} \right)^{1/n} \quad .$$

Then $|z| > \max(N_0,\, N_1)$ implies $|p(z)| > M$. This proves the limit formula.

To see that p is proper, suppose that K is a compact subset of $\mathbf{F}$, and choose $M > 0$ such that $w \in K$ implies $|w| \leq M$. Let N be the maximum of N_1 and N_2, where these are defined as in the preceding paragraph. We then have that the closed set $p^{-1}[K]$ lies in the bounded set of points satisfying $|z| \leq N$, and therefore $p^{-1}[K]$ is a compact subset of $\mathbf{F}$.■

4. As usual, we follow the hints.

> *[Show that] the family $\{ A_n \}$ is a locally finite family of closed locally compact subspaces in A.*

If $(x, t) \in \ell^2 \times (0, +\infty)$ and U is the open set $\ell^2 \times (t/2,\, +\infty)$, then $x \in U$ and $A_n \cap U = \emptyset$ unless $2^{-n} \geq t/2$, and therefore the family is locally finite. Furthermore, each set is closed in $\ell^2 \times (0, +\infty)$ and therefore also closed in $A = \bigcup_n A_n$.

> *Use this to show that the union is locally compact.*

We shall show that *if A is a $\mathbf{T_3}$ space that is a union of a locally finite family of closed locally compact subsets A_α, then A is locally compact.* Let $x \in A$, and let U be an open subset of A containing x such that $U \cap A_\alpha = \emptyset$ unless $\alpha = \alpha_1 \ \cdots \ , \alpha_k$. Let V_0 be an open subset of A such that $x \in V_0 \subset \overline{V_0} \subset U$, for each i choose an open set $W_i \in A_{\alpha_i}$ such that the closure of W_i is compact, express W_i as an intersection $V_i \cap A_{\alpha_i}$ where V_i is open in A, and finally let $V = \cap_i V_i$. Then we have

$$\overline{V} \quad = \quad \bigcup_{i=1}^{k} \left(\overline{V} \cap A_{\alpha_i}\right) \quad \subset \quad \bigcup_{i=1}^{k} \left(\overline{V_i} \cap A_{\alpha_i}\right) \quad = \quad \bigcup_{i=1}^{k} \textbf{Closure}\, (W_i \ \text{ in } \ A_{\alpha_i}) \ .$$

Since the set on the right hand side is compact, the same is true for $\overline{V}$. Therefore we have shown that A is locally compact.

> *Show that the closure of A contains all of $\ell^2 \times \{0\}$. Explain why ℓ^2 is not locally compact.*

Let $x \in \ell^2$, and for each positive integer k let $P_k(x) \in A_k$ be the point $(H_k(x), 2^{-(k+1)}\,)$, where $H_k(x)$ is the point whose first k coordinates are those of x and whose remaining coordinates are 0. It is an elementary exercise to verify that $(x, 0) = \lim_{k\to\infty} P_k(x)$. To conclude we need to show that ℓ^2 is not locally compact. If it were, then there would be some $\varepsilon > 0$ such that the set of all $y \in \ell^2$ satisfying every $|y| \leq \varepsilon$ would be compact, and consequently infinite sequence $\{ y_n \}$ in ℓ^2

with $|y_n| \leq \varepsilon$ (for all n) would have a convergent subsequence. To see this does not happen, let $y_k = \frac{1}{2}\varepsilon\, \mathbf{e}_k$, where $\mathbf{e}_k$ is the $k^{\rm th}$ standard unit vector in ℓ^2. This sequence satisfies the boundedness condition but does not have a convergent subsequence. Therefore ℓ^2 is not locally compact.■

FOOTNOTE. Basic theorems from functional analysis imply that a normed vector space is locally compact if and only if it is finite-dimensional.■

5. Suppose that V is open in $U^\bullet$. If $\infty_U \notin V$ then $V \subset U$ and $c^{-1}[V] = V$, so the set on the left hand side of the equation is open. Suppose now that $\infty_U \in V$; then $A = U^\bullet - V$ is a compact subset of U and $c^{-1}[V] = X - c^{-1}[A] = X - A$, which is open because the compact set A is also closed in X.

There are many examples for which c is not open. For example, let $X = [0,5]$ and $A = [1,3]$; in this example the image J of the open set $(2,4)$ is not open because the inverse image of J is $[1,4)$, which is not open. More generally, if X is connected and $X - U$ has a nonempty interior, then $X \to U^\bullet$ is not open (try to prove this!).■

FOOTNOTE. In fact, if $F = X - U$ then $U^\bullet$ is homeomorphic to the space X/F described in a previous exercise. This is true because the collapse map passes to a continuous map from X/F to $U^\bullet$ that is 1–1 onto, and this map is a homeomorphism because X/F is compact and $U^\bullet$ is Hausdorff.■

6. (a) If X is compact Hausdorff and $f : X \to Y$ is a continuous map into a Hausdorff space, then $f[X]$ is closed. Therefore $f[X] = Y$ if the image of f is dense, and in fact f is a homeomorphism.■

(b) This exercise shows that a formal analog of an important concept (the Stone-Čech compactification) is not necessarily as useful as the original concept itself; of course, there are also many situations in mathematics where the exact opposite happens. In any case, given an abstract closure (Y, f) we must have $(X, \mathrm{id}_X) \geq (Y, f)$ because $f : X \to Y$ trivially satisfies the condition $f = f \circ \mathrm{id}_X$.■

7. See the footnote to Exercise 5 above.■

8. Suppose that X is uniformly locally compact as above and $\{x_n\}$ is a Cauchy sequence in X. Choose M such that $m, n \geq M$ implies $\mathbf{d}(x_m, x_n) < \delta$, where δ is as in the problem. If we define a new sequence by $y_k = x_{k-M}$, then $\{y_k\}$ is a Cauchy sequence whose values lie in the (compact) closure of $N_\delta(x_M)$, and since compact metric spaces are complete the sequence $\{y_k\}$ has some limit z. Since $\{y_k\}$ is just $\{x_{k-M}\}$, it follows that the sequence $\{x_{k-M}\}$, and hence also the sequence $\{x_n\}$, must converge to the same limit z.

One of the simplest examples of a noncomplete metric space with the weaker property is the half-open interval $(0, \infty)$. In this case, if we are given x we can take $\delta_x = x/2$.■

9. Let $B = A \cup \{\infty_X\}$, and consider the function $g : A^\bullet \to X$ such that g is the inclusion on A and g sends ∞_A to ∞_X. By construction this map is continuous at all ordinary points of A, and if g is also continuous at ∞_A then g will define a 1–1 continuous map from $A^\bullet$ onto B, and this map will be a homeomorphism.

Suppose now that V is an open neighborhood of ∞_X in $X^\bullet$. By definition we know that $K = X - V$ is compact. But now we have

$$g^{-1}[V] \;=\; A^\bullet \;-\; g^{-1}[X - V] \;=\; A^\bullet \;-\; A \cap K\ .$$

Now $A \cap K$ is compact because both A and $X - V$ are closed X and K is a compact subset of the Hausdorff space X, and therefore it follows that the set $g^{-1}[V] \;=\; A^\bullet \;-\; A \cap K$ is open in A, so that g is continuous everywhere.■

10. Follow the hint. The final sentence is equivalent to the desired conclusion because an arbitrary open neighborhood of ∞ has the form $\{\infty\} \cup (\mathbb{C} - K)$, where K is a compact subset of $\mathbb{C}$. Since the compact set K is bounded in $\mathbb{C}$ we can find some $M > 0$ such that all points of K have distance strictly less than M from the origin. Let $\varepsilon = 1/M$, and using the limit assumption for $1/f$ at a let δ be chosen so that $0 < |z - a| < \delta$ implies $|1/f(z)| < \varepsilon$. Then the given conditions on z also imply $|f(z)| > M$, so that $f(z) \notin K$, which is what we needed to show.■

VI.5 : Metrization theorems

Problems from Munkres, § 40, p. 252

2. If W is an $\mathbf{F}_\sigma$ set then $W = \cup_n F_n$ where n ranges over the nonnegative integers and F_n is closed in X. Therefore

$$X - W \;=\; \bigcap_n X - F_n$$

is a countable intersection of the open subsets $X - F_n$ and accordingly is a $\mathbf{G}_\delta$ set.

Conversely, if $V = X - W$ is a $\mathbf{G}_\delta$ set, then $V = \cap_n U_n$ where n ranges over the nonnegative integers and U_n is open in X. Therefore

$$W \;=\; X - V \;=\; \bigcup_n X - U_n$$

is a countable union of the closed subsets $X - U_n$ and accordingly is an $\mathbf{F}_\sigma$ set.■

FOOTNOTE. We have already shown that a closed subset of a metrizable space is a $\mathbf{G}_\delta$ set, and it follows that every open subset of a metrizable space is an $\mathbf{F}_\sigma$ set.■

3. Suppose that X is $\mathbf{T_1}$ and has a locally finite base $\mathcal{B}$. Then for each $x \in X$ there is an open set W_x containing x such that $W_x \cap V_\beta = \emptyset$ for all β except $\beta(1), \cdots, \beta(k)$. Let V^* be the intersection of all sets in the finite subcollection that contain x. Since $\mathcal{B}$ is a base for this topology it follows that some $V_{\beta(J)}$ contains x and is contained in this intersection, But this means that $V_{\beta(J)}$ must be contained in all the other open subsets in $\mathcal{B}$ that contain x and is therefore a *minimal* open subset containing x. Suppose this minimal open set has more than one point; let y be another point in the set. Since X is $\mathbf{T_1}$ it will follow that $V_{\beta(J)} - \{y\}$ is also an open subset containing x. However, this contradicts the minimality of $V_{\beta(J)}$ and shows that the latter consists only of the point $\{x\}$. Since x was arbitrary, this shows that every one point subset of X is open and thus that X is discrete.■

Additional exercises

1. (a) None of the arguments verifying the axioms for open sets in metric spaces rely on the assumption $\mathbf{d}(x,y) = 0 \Longrightarrow x = y$.■

(b) The relation is clearly reflexive and symmetric. To see that it is transitive, note that $\mathbf{d}(x,y) = \mathbf{d}(y,z) = 0$ and the Triangle Inequality imply

$$\mathbf{d}(x,z) \;\leq\; \mathbf{d}(x,y) + \mathbf{d}(y,z) \;=\; 0 + 0 \;=\; 0\,.$$

Likewise, if $x \sim x'$ and $y \sim y'$ then

$$\mathbf{d}(x',y') \;\leq\; \mathbf{d}(x',x) + \mathbf{d}(x,y) + \mathbf{d}(y,y') \;=\; 0 + \mathbf{d}(x,y) + 0 \;=\; \mathbf{d}(x,y)$$

and therefore the distance between two points only depends upon their equivalence classes with respect to the given relation.■

(c) First of all we verify that $\mathbf{d}_\infty$ defines a metric. In order to do this we must use some basic properties of the function

$$\varphi(x) \quad = \quad \frac{x}{1+x} \ .$$

This is a continuous and strictly increasing function defined on $[0, +\infty)$ and taking values in $[0,1)$ and it has the additional property $\varphi(x+y) \leq \varphi(x) + \varphi(y)$. The continuity and monotonicity properties of φ follow immediately from a computation of its derivative, the statement about its image follows because $x \geq 0$ implies $0 \leq \varphi(x) < 1$ and $\lim_{x\to+\infty} \varphi(x) = 1$ (both calculations are elementary exercises that are left to the reader).

The inequality $\varphi(x+y) \leq \varphi(x) + \varphi(y)$ is established by direct computation of the difference $\varphi(x) + \varphi(y) - \varphi(x+y)$:

$$\frac{x}{1+x} + \frac{y}{1+y} - \frac{x+y}{1+x+y} \quad = \quad \frac{x^2 y \ + \ 2\,xy \ + \ xy^2}{(1+x)\,(1+y)\,(1+x+y)}$$

This expression is nonnegative if x and y are nonnegative, and therefore one has the desired inequality for φ. Another elementary but useful inequality is $\varphi(x) \leq x$ if $x \geq 0$ (this is true because $1 \leq 1 + x$). Finally, we note that the inverse to the continuous strictly monotonic function φ is given by

$$\varphi^{-1}(y) \quad = \quad \frac{y}{1-y} \ .$$

It follows that if $\mathbf{d}$ is a pseudometric then so is $\varphi \circ \mathbf{d}$ with the additional property that $\varphi \circ \mathbf{d} \leq 1$. More generally, if $\{ a_n \}$ is a convergent sequence of nonnegative real numbers and $\{ \mathbf{d}_n \}$ is a sequence of pseudometrics on a set X, then

$$\mathbf{d}_\infty \quad = \quad \sum_{n=1}^{\infty} a_n \cdot \varphi \circ \mathbf{d}_n < \sum_{n=1}^{\infty} a_n < \infty$$

also defines a pseudometric on X (write out the details of this!). In our situation $a_n = 2^{-n}$. Therefore the only thing left to prove about $\mathbf{d}_\infty$ is that it is positive when $x \neq y$. But in our situation if $x \neq y$ then there is some n such that $\mathbf{d}_n(x,y) > 0$, and the latter in turn implies that

$$2^{-n}\varphi\big(\,\mathbf{d}_n(x,y)\,\big) \quad > \quad 0$$

and since the latter is one summand in the infinite sum of nonnegative real numbers given by $\mathbf{d}_\infty(x,y)$ it follows that the latter is also positive. *Therefore* $\mathbf{d}_\infty$ *defines a metric on* X.

To prove that the topology $\mathcal{M}$ defined by this metric and the topology $\mathbf{T}_\infty$ determined by the sequence of pseudometrics are the same. Let N_α denote an α-neighborhood with respect to the $\mathbf{d}_\infty$ metric, and for each n let $N^{\langle n \rangle}_\beta$ denote a β-neighborhood with respect to the pseudometric $\mathbf{d}_n$. Suppose that N_ε is a basic open subset for $\mathcal{M}$ where $\varepsilon > 0$ and $x \in X$. Choose A so large that $n \geq A$ implies

$$\sum_{k=A}^{\infty} 2^{-k} \quad < \quad \frac{\varepsilon}{2}$$

Let W_x be the set of all z such that $\mathbf{d}_k(x,z) < \varepsilon/2$ for $1 \leq k < A$. Then W_x is the finite intersection of the $\mathbf{T}_\infty$-open subsets

$$W^{\langle k \rangle}(x) \quad = \quad \{ \ z \in X \ \ | \ \ \mathbf{d}_k(x,z) < \varepsilon/2 \ \}$$

and therefore W_k is also $\mathbf{T}_\infty$-open. Direct computation shows that if $y \in W_k$ then

$$\mathbf{d}_\infty(x,y) \;=\; \sum_{n=1}^{\infty} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,) \;=\; \sum_{n=1}^{A-1} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,) \;+\; \sum_{n=A}^{\infty} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,) \;<$$

$$\left(\sum_{n=1}^{A-1} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,)\right) + \frac{\varepsilon}{2} \;<\; \left(\sum_{n=1}^{A-1} \frac{2^{-n}\,\varepsilon}{2}\right) + \frac{\varepsilon}{2} \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;<\; \varepsilon$$

so that $W_x \subset N_\varepsilon(x)$. Therefore, if U is open with respect to $\mathbf{d}_\infty$ we have

$$U \;=\; \bigcup_{x\in U} \{x\} \;\subset\; \bigcup_{x\in U} W_x \;\subset\; \bigcup_{x\in U} N_\varepsilon(x) \;\subset\; U$$

which shows that U is a union of $\mathbf{T}_\infty$-open subsets and therefore is $\mathbf{T}_\infty$-open. *Thus $\mathcal{M}$ is contained in $\mathbf{T}_\infty$.*

To show the reverse inclusion, consider an subbasic $\mathbf{T}_\infty$-open subset of the form $N_\varepsilon^{\langle n\rangle}(x)$. If y belongs to the latter then there is a $\delta > 0$ such that $N_\delta^{\langle n\rangle}(y) \subset N_\varepsilon^{\langle n\rangle}(x)$; without loss of generality we may as well assume that $\delta < 2^{-k}$. If we set

$$\eta(y) \;=\; 2^{-k}\,\varphi\big(\,\delta(y)\,\big)$$

then $\mathbf{d}_\infty(z,y) < \eta(y)$ and

$$2^{-k}\,\varphi\circ\mathbf{d}_k \;\leq\; \mathbf{d}_\infty$$

imply that $2^{-k}\,\varphi\circ\mathbf{d}_k(z,y) < \eta(y)$ so that $\varphi\circ\mathbf{d}_k(z,y) < 2^k\,\eta(y)$ and

$$\mathbf{d}_k(z,y) \;<\; \varphi^{-1}\big(\,2^k\,\delta(y)\,\big)$$

and by the definition of $\eta(y)$ the right hand side of this equation is equal to $\delta(y)$. Therefore if we set $W = N_\varepsilon^{\langle k\rangle}(x)$ then we have

$$W \;=\; N_\varepsilon^{\langle k\rangle}(x) \;=\; \bigcup_{y\in W} \{y\} \;\subset\; \bigcup_{y\in W} N_{\eta(y)}(y) \;\subset\; \bigcup_{y\in W} N_{\delta(y)}^{\langle k\rangle}(y) \;\subset\; N_\varepsilon^{\langle k\rangle}(x)$$

which shows that $N_\varepsilon^{\langle k\rangle}(x)$ belongs to $\mathcal{M}$. *Therefore the topologies $\mathbf{T}_\infty$ and $\mathcal{M}$ are equal.*■

(d) Take the pseudometrics $\mathbf{d}_n$ as in the hint, and given $h \in X$ let $J^{\langle n\rangle}(h)$ be its restriction to $[-n,n]$. Furthermore, let $\|...\|_n$ be the uniform metric on $\mathbf{BC}\,(\,[-n,n]\,)$, so that

$$\mathbf{d}_n(f,g) \;=\; \|\,J^{\langle n\rangle}(f) \;-\; J^{\langle n\rangle}(g)\,\|_n\ .$$

Given $\varepsilon > 0$ such that $\varepsilon < 1$, let $\delta = 2^{-n}\,\varphi(\varepsilon)$. Then $\mathbf{d}_\infty(f,g) < \delta$ implies $2^{-n}\,\varphi\circ\mathbf{d}_n(f,g) < \delta$, which in turn implies $\mathbf{d}_n(f,g) < \varphi^{-1}(2^n\,\delta) = \varepsilon$.■

(e) We first claim that $\lim_{n\to\infty} \mathbf{d}_\infty(f_n,f) = 0$ if and only if $\lim_{n\to\infty} \mathbf{d}_k(f_n,f) = 0$ for all k.

($\Longrightarrow$) Let $\varepsilon > 0$ and fix k. Since $\lim_{n\to\infty} \mathbf{d}_\infty(f_n,f) = 0$ there is a positive integer M such that $n \geq M$ implies $\mathbf{d}_\infty(f_n,f) < 2^k\,\varphi(\varepsilon)$. Since

$$2^{-k}\varphi\circ\mathbf{d}_k \;\leq\; \mathbf{d}_\infty$$

it follows that $\mathbf{d}_k \leq \varphi^{-1}(2^{-k}\,\mathbf{d}_\infty)$ and hence that $\mathbf{d}_k(f_n, f) < \varepsilon$ if $n \geq M$.■

($\Longleftarrow$) Let $\varepsilon > 0$ and choose A such that

$$\sum_{k=n}^{\infty} 2^{-k} \quad < \quad \frac{\varepsilon}{2}\,.$$

Now choose B so that $n \geq B$ implies that

$$\mathbf{d}_k(f_n, f) \quad < \quad \frac{\varepsilon}{2A}$$

for all $k < A$. Then if $n \geq A + B$ we have

$$\mathbf{d}_\infty(f_n, f) \quad = \quad \sum_{k=1}^{\infty} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \quad = \quad \sum_{k=1}^{A} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \;+\; \sum_{k=A}^{\infty} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \quad <$$

$$\sum_{k=1}^{A} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \;+\; \frac{\varepsilon}{2} \quad < \quad \left(\sum_{k=1}^{A} 2^{-k}\frac{\varepsilon}{2A}\right) \;+\; \frac{\varepsilon}{2} \quad = \quad \varepsilon$$

so that $\lim_{n\to\infty} \mathbf{d}_\infty(f_n, f) = 0$.■

Now suppose that K is a compact subset of $\mathbb{R}$, and let $\|...\|_K$ be the uniform norm on $\mathbf{BC}(K)$. Then $K \subset [-n, n]$ for some n and thus for all $g \in X$ we have $\|g\|_K \leq \mathbf{d}_n(g, 0)$. Therefore if $\{\, f_n \,\}$ converges to f then the sequence of restricted functions $\{\, f_n|K \,\}$ converges uniformly to $f|K$. Conversely, if for each compact subset $K \subset \mathbb{R}$ the sequence of restricted functions $\{\, f_n|K \,\}$ converges to $f|K$, then this is true in particular for $K = [-L, L]$ and accordingly $\lim_{n\to\infty} \mathbf{d}_L(f_n, f) = 0$ for all L. However, as noted above this implies that $\lim_{n\to\infty} \mathbf{d}_\infty(f_n, f) = 0$ and hence that $\{\, f_n \,\}$ converges to f.■

(f) The answer is **YES**, and here is a proof: Let $\{\, f_n \,\}$ be a Cauchy sequence in X. Then if $K_m = [-m, m]$ the sequence of restricted functions $\{\, f_n|K_m \,\}$ is a Cauchy sequence in $\mathbf{BC}(K_m)$ and therefore converges to a limit function $g_m \in \mathbf{BC}(K_m)$. Since $\lim_{n\to\infty} f_n|K_m(x) = g_m(x)$ for all $x \in K_m$ it follows that $p \leq m$ implies $g_m|K_p = g_p$ for all such m and p. Therefore if we define $g(x) = g_m(x)$ if $|x| < m$ then the definition does not depend upon the choice of m and the continuity of g_m for each m implies the continuity of g. Furthermore, by construction it follows that

$$\lim_{n\to\infty} \mathbf{d}_m(f_n, g) \quad = \quad \lim_{n\to\infty} \|\,(f_n|K_m) \;-\; g_m\,\| \quad = \quad 0$$

for all m and hence that $\lim_{n\to\infty} \mathbf{d}_\infty(f_n, f) = 0$ by part **(e)** above.■

(g) The key idea is to express U as an increasing union of bounded open subsets V_n such that $\overline{V_n} \subset V_{n+1}$ for all n. If U is a proper open subset of $\mathbb{R}^n$ let $F = \mathbb{R}^n - U$ (hence F is closed), and let V_m be the set of all points x such that $|x| < m$ and $\mathbf{d}_{[\mathbf{2}]}(x, F_m) > 1/m$, where $\mathbf{d}_{[\mathbf{2}]}$ denotes the usual Euclidean metric; if $U = \mathbb{R}^n$ let V_m be the set of all points x such that $|x| < m$. Since $y \in U$ if and only if $\mathbf{d}_{[\mathbf{2}]}(y, F) = 0$, it follows that $U = \cup_m V_m$. Furthermore, since $y \in \overline{V_m}$ implies $|x| \leq m$ and $\mathbf{d}_{[\mathbf{2}]}(x, F_n) \geq 1/n$ (why?), we have $\overline{V_m} \subset V_{m+1}$. Since $\overline{V_m}$ is bounded it is compact.

If f and g are continuous real valued functions on U, define $\mathbf{d}_n(f, g)$ to be the maximum value of $|f(x) - g(x)|$ on $\overline{V_m}$. In this setting the conclusions of parts **(d)** through **(f)** go through with only one significant modification; namely, one needs to check that every compact subset of U is contained in some V_m. To see this, note that K is a compact subset that is disjoint from the closed subset F, and therefore the continuous function $\mathbf{d}_{[\mathbf{2}]}(y, F)$ assumes a positive minimum value c_1 on K and that there is a positive constant c_2 such that $y \in K$ implies $|y| \leq c_2$. If we choose m such that $m > 1/c_1$ and $m > c_2$, then K will be contained in V_m as required.■

FOOTNOTE. Here is another situation where one encounters metrics defined by an infinite sequence of pseudometrics. Let Y be the set of all infinitely differentiable functions on $[0,1]$, let D^k denote the operation of taking the k^{th} derivative, and let $\mathbf{d}_k(f,g)$ be the maximum value of $|D^k f - D^k g|$, where $0 \leq k < \infty$. One can also mix this sort of example with the one studied in the exercise; for instance, one can consider the topology on the set of infinitely differentiable functions on $\mathbb{R}$ defined by the countable family of pseudometrics $\mathbf{d}_{k,n}$ where $\mathbf{d}_{k,n}(f,g)$ is the maximum of $|D^k f - D^k g|$ on the closed interval $[-n, n]$.■

2. (a) We shall follow the steps indicated in the hint(s).

Note that

$$(\mathbb{R}^n)^\bullet - \{0\} \cong \mathbb{R}^n$$

This is true because the left hand side is homeomorphic to the complement of a point in S^n, and such a complement is homeomorphic to $\mathbb{R}^n$ via stereographic projection (which may be taken with respect to an arbitrary unit vector on the sphere).■

Consider the continuous function on

$$\left(\overline{V} - V\right) \sqcup \{\infty\} \subset (\mathbb{R}^n)^\bullet - \{0\}$$

defined on the respective pieces by the restriction of f and ∞. Why can this be extended to a continuous function on $(\mathbb{R}^n)^\bullet - V$ with the same codomain?

In the first step we noted that the codomain was homeomorphic to $\mathbb{R}^n$, and the Tietze Extension Theorem implies that a continuous function from a closed subset A of a metric space X into $\mathbb{R}^n$ extends to all of X.■

What happens if we try to piece this together with the original function f defined on U?

If h is the function defined above, then we can piece h and $f|\overline{V}$ together and obtain a continuous function on all of $(\mathbb{R}^n)^\bullet$ if and only if the given functions agree on the intersection of the two closed subsets. This intersection is equal to $\overline{V} - V$, and by construction the restriction of h to this subset is equal to the restriction of f to this subset.■

(b) Let h and h' denote the associated maps from $(\mathbb{R}^n)^\bullet - V$ to $(\mathbb{R}^n)^\bullet - \{0\} \cong \mathbb{R}^n$ that are essentially given by the restrictions of g and g'. Each of these maps has the same restriction to $\left(\overline{V} - V\right) \sqcup \{\infty\}$. Define a continuous mapping $H : ((\mathbb{R}^n)^\bullet - V) \times [0,1] \longrightarrow (\mathbb{R}^n)^\bullet - \{0\} \cong \mathbb{R}^n$ by $H(x,t) = t\,h'(x) + (1-t)\,h(x)$, and define F' on $\overline{V} \times [0,1]$ by $F(x,t) = f(x)$. As in the first part of this exercise, the mappings H and F' agree on the intersection of their domains and therefore they define a continuous map G on all of $S^n \times [0,1]$. Verification that $G(x,0) = g(x)$ and $G(x,1) = g'(x)$ is an elementary exercise.■

3. (a) Let U be open in X, and let $F = X - U$. Then by the examples cited above we know that F is a closed set which is also a G_δ set. Apply Exercise 2 on page 252 to show that $U = X - F$ must be an open F_σ set.■

(b) If A is the zero set of a continuous function as above, then

$$A = \cap_n \geq 1\ f^{-1}\left[\,[0, 1/n)\,\right]$$

where each half open interval $[0, n)$ is open in $[0, 1]$, so that their inverse images are open in X. Conversely, if A is a closed G_δ of the form $\cap_n U_n$, then if we set $V_n = U_1 \cap \cdots \cap U_n$, we have $A = \cap_n V_n$ and $V_1 \supset V_2 \supset \cdots$. Define functgions $g_n : X \to [0, 1]$ such that $g_n = 0$ on A and $g_n = 1$ on $F_n = X - V_n$. If we take $f = \sum_n 2^{-n} \cdot g_n$, then the infinite series converges absolutely and uniformly to a continuous function, and the zero set of this function is precisely A.■

(c) Let $A = X - W$, and let f be as in the previous part of this exercise. Then the set of zero points is A, so the set of points where f is nonzero (and in fact positive by construction) will be the set $W = X - A$.■

4. For each x let φ_x be a continuous real valued function in $\mathcal{F}$ such that $\varphi_x = 0$ on some open set U_x containing x. The family of open sets $\{ U_x \}$ is an open covering of X and therefore has a finite subcovering by sets $U_{x(i)}$ for $1 \leq i \leq k$. The product of the functions $\prod_i f_{x(i)}$ belongs to $\mathcal{F}$ and is zero on each $U_{x(i)}$; since the latter sets cover X it follows that the product is the zero function and therefore the latter belongs to $\mathcal{F}$.■

5. REMINDER. If X is a topological space and $\mathcal{U} = \{U_1, \cdots, U_n\}$ is a finite indexed open covering of X, then a partition of unity subordinate to $\mathcal{U}$ is an indexed family of continuous functions $\varphi_i : X \to [0, 1]$ for $1 \leq i \leq n$ such that for each i the zero set of the function φ_i contains contains $X - U_i$ in its interior and

$$\sum_{i=1}^{n} \varphi_i \quad = \quad 1 \; .$$

Theorem 36.1 on pages 225–226 of Munkres states that for each finite indexed open covering $\mathcal{U}$ of a $\mathbf{T_4}$ space (hence for each such covering of a compact Hausdorff space), there is a partition of unity subordinate to $\mathcal{U}$. The proof of this is based upon Urysohn's Lemma, so the methods in Munkres can be combined with our proof of the result for metric space to prove the existence of partitions of unity for indexed finite open coverings of compact metric spaces.

The solution to the exercise now proceeds as follows:

For each $x \in X$ we are assuming the existence of a continuous bounded function f_x such that $f_x(x) \neq 0$. Since J is closed under multiplication, we may replace this function by it square if necessary to obtain a function $g_x \in J$ such that $g_x(x) > 0$. Let U_x be the open set where g_x is nonzero, and choose an open subset V_x such that $\overline{V_x} \subset U_x$. The sets V_x determine an open covering of X; this open covering has a finite subcovering that we index and write as $\mathcal{V} = \{V_1, \cdots, V_n\}$. For each i let $g_i \in J$ be the previously chosen function that is positive on $\overline{V_i}$, and consider the function

$$g \quad = \quad \sum_{i=0}^{k} \varphi_i \cdot g_i \; .$$

This function belongs to J, and we claim that $g(y) > 0$ for all $y \in X$. Since $\sum_i \varphi_i = 1$ there is some index value m such that $\varphi_m(y) > 0$; by definition of a partition of unity this means that $y \in V_m$. But $y \in V_m$ implies $g_m(y) > 0$ too and therefore we have $g(y) \geq \varphi_m(y)\, g_m(y) > 0$. Since the reciprocal of a nowhere vanishing continuous real valued function on a compact space is continuous (and bounded!), we know that $1/g$ lies in $\mathbf{BC}(X)$. Since $g \in J$ it follows that $1 = g \cdot (1/g)$ also lies in J, and this in turn implies that $h = h \cdot 1$ lies in J for all $h \in \mathbf{BC}(X)$. Therefore $J = \mathbf{BC}(X)$ as claimed.■

6. Let $\mathcal{M}$ be the set of all maximal ideals. For each point $x \in X$ we need to show that $\mathbf{M}_x \in \mathcal{M}$. First of all, verification that $\mathbf{M}_x$ is an ideal is a sequence of elementary computations (which the reader should verify). To see that the ideal is maximal, consider the function $\widehat{x} : \mathbf{BC}(X) \to \mathbb{R}$ by the formula $\widehat{x}(f) = f(x)$. This mapping is a ring homomorphism, it is onto, and $\widehat{x}(f) = 0$ if and only if $f \in \mathbf{M}_x$. Suppose that the ideal is not maximal, and let $\mathbf{A}$ be an ideal such that $\mathbf{M}_x$ is properly contained in $\mathbf{A}$. Let $a \in A$ be an element that is not in $\mathbf{M}_x$. Then $a(x) = \alpha \neq 0$ and it follows that $a(x) - \alpha 1$ lies in $\mathbf{M}_x$. It follows that $\alpha 1 \in \mathbf{A}$, and since $\mathbf{A}$ is an ideal we also have that $1 = \alpha^{-1}(\alpha 1)$ lies in $\mathbf{A}$; the latter in turn implies that every element $f = f \cdot 1$ of $\mathbf{BC}(X)$ lies in $\mathbf{A}$.

We claim that the map from X to $\mathcal{M}$ sending x to $\mathbf{M}_x$ is 1–1 and onto. Given distinct points x and y there is a bounded continuous function f such that $f(x) = 0$ and $f(y) = 1$, and therefore it follows that $\mathbf{M}_x \neq \mathbf{M}_y$. To see that the map is onto, let $\mathbf{M}$ be a maximal ideal, and note that the preceding exercise implies the existence of some point $p \in X$ such that $f(p) = 0$ for all $f \in \mathbf{M}$. This immediately implies that $\mathbf{M} \subset \mathbf{M}_p$, and since both are maximal (proper) ideals it follows that they must be equal. Therefore the map from X to $\mathcal{M}$ is a 1–1 correspondence.■

FINAL FOOTNOTES. **(1)** One can use techniques from functional analysis and Tychonoff's Theorem to put a natural topology on $\mathcal{M}$ (depending only on the structure of $\mathbf{BC}(X)$ as a Banach space and an algebra over the reals) such that the correspondence above is a homeomorphism; see page 283 of Rudin, *Functional Analysis*, for more information about this.

(2) The preceding results are the first steps in the proof of an important result due to I. Gelfand and M. Naimark that give a complete set of abstract conditions under which a Banach algebra is isomorphic to the algebra of continuous complex valued functions on a compact Hausdorff space. A Banach algebra is a combination of Banach space and associative algebra (over the real or complex numbers) such that the multiplication and norm satisfy the compatibility relation $|xy| \leq |x| \cdot |y|$. The additional conditions required to prove that a Banach algebra over the complex numbers is isomorphic to the complex version of $\mathbf{BC}(X)$ are commutativity, the existence of a unit element, and the existence of an conjugation-like map (formally, an **involution**) $a \to a^*$ satisfying the additional condition $|aa^*| = |a|^2$. Details appear in Rudin's book on functional analysis, and a reference for the Gelfand-Naimark Theorem is Theorem 11.18 on page 289. A classic reference on Banach algebras (definitely NOT up-to-date in terms of current knowledge but an excellent source for the material it covers) is the book by Rickart in the bibliographic section of the course notes.