SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 6

Fall 2014

APPENDICES

Appendix A : Topological groups

These exercises are taken from various sections in Munkres. Some of these solutions are based upon material in the site `http://dbfin.com/topics/math/`.

Supplementary Exercises from Munkres, pp. 145 – 146

5. (a) Let $\pi : G \to G/H$ be the quotient projection. If $L_a : G \to G$ is the left multiplication homeomorphism for $a \in G$, then by associativity of multiplication we know that $\pi(x) = \pi(y)$ (equivalently, $xH = yH$) implies that $\pi \circ L_a(x) = axH = ayH = \pi \circ L_a(y)$, and therefore by the properties of quotient topologies there is a unique map $L'_a : G/H \to G/H$ such that $L'_a \circ \pi = \pi \circ \mathrm{L}_a$. If $a = 1$, so that L_a is the identity, then the unique mapping L'_a must also be the identity; furthermore, the uniqueness properties and $L_{ba} = L_b \circ L_a$ imply that $L'_{ba} = L'_b \circ L'_a$. If we now let $\{a, b\} = \{c, c^{-1}\}$ for some $c \in G$, the conclusions of the previous sentence imply that L'_c is a homeomorphism and $(L'_c)^{-1} = L'_{c^{-1}}$.

Suppose now that we have cosets xH and yH. If $a = yx^{-1}$, then by construction L'_a sends xH to yH, and therefore G/H is homogeneous.■

WARNING. In the overwhelming majority of mathematical writings over the past several decades, the phrase *homogeneous space* is reserved for the sorts of coset spaces we have described here. Normaly mathematicians would use the phrase "space which is homogeneous" for the spaces described as "homogeneous spaces" in Munkres.

(b) If H is a closed subgroup, then by the definition of the quotient topology the subset $\{H\} \subset G/H$ is closed (more generally, a one point subset in a quotient space is closed if and only if its equivalence class in the original space is closed).

More generally, if Y is a topological space such that $\{y\}$ is closed for some $y \in Y$, then for every homeomorphism $h : Y \to Y$ we know that $h[\{y\}] = \{h(y)\}$ is also a closed subset. Specializing to G/H, if $xH \in G/H$ then we know that there is a homeomorphism from G/H to itself sending H to xH by (a). If H is closed in G, then we know that $\{H\} \subset G/H$ is closed, and therefore it follows that $\{xH\}$ is also closed.■

(c) Suppose that U is open in G. Then $\pi[U]$ is the union of all cosets uH where u runs through the elements of U, and we also have $\pi^{-1}[\pi[U]] = \cup_{h \in H} U \cdot h$. Since right multiplication by h is a homeomorphism and U is open in G, it follows that each $U \cdot h$ is also open in G and hence their union is also open in G. By the definition of the quotient topology, this means that $\pi[U]$ is open in G/H.■

(d) We shall modify this part as follows: If H is a normal subgroup of G, then G/H is a topological group in the sense of the course notes (no assumption that $\{1\}$ is a closed subset), and if H is closed then this topological group is a $\mathbf{T}_1$ space (and hence a topological group in the sense of Munkres).

If G is a topological group and H is a normal subgroup then by standard results on groups we know that the group structure on G passes to a group structure on G/H. In terms of commutative diagrams, if m denotes the multiplication on G and χ denotes the inverse map on G, then these and the corresponding maps m' and χ' for G/H satisfy the following identities:

$$\pi^\circ m \;=\; m'^\circ(\pi \times \pi)\,, \qquad\qquad \pi^\circ\chi \;=\; \chi'^\circ\pi$$

We need to show that m' and χ' are continuous with respect to the quotient topology on G/H. The continuity of χ' follows immediately from the identity because π is a quotient map, but as noted in the next paragraph, the continuity of m' requires some further discussion.

If we impose the quotient topology on $G/H \times G/H$ associated to the continuous mapping $\pi \times \pi$, then general considerations imply that m' is continuous. However, the topology we really want on $G/H \times G/H$ is the self-product of the quotient topology for π. We need to verify that these two topologies are equal. If they are, then it will follow that the quotient group structure on G/H makes the latter (with the quotient topology) into a topological group.

We can formulate the issue more abstractly as follows: Suppose that we are given a topological space $(X, \mathbf{T}_X)$ and an onto map of sets $p : X \to Y$. Denote the quotient topology on Y by $p_*\mathbf{T}_X$. The product map $p \times p : X \times X \to Y \times Y$ is also onto, and thus we also have the quotient topology $(p \times p)_*\mathbf{T}_{X\times X}$ on $Y \times Y$. We need to show that, at least in some cases, the spaces $(Y \times Y, (p \times p)_*\mathbf{T}_{X\times X}$ and $(Y, p_*\mathbf{T}_X) \times (Y, p_*\mathbf{T}_X)$ are identical.

> CLAIM. *The statement in the preceding sentence is valid if there is a topology* $\mathbf{U}$ *on* Y *such that* $p : (X, \mathbf{T}_X) \to (Y, \mathbf{U})$ *is open.* — The openness hypothesis implies that $\mathbf{U} = p_*\mathbf{T}_X$, and the product map $p \times p : (X, \mathbf{T}_X) \times (X, \mathbf{T}_X) \to (Y, \mathbf{U}) \times (Y, \mathbf{U})$ is also open (since $p \times p$ sends basic open sets to basic open sets). Therefore $p \times p$ is a quotient map, and this implies that the spaces $(Y \times Y, (p \times p)_*\mathbf{T}_{X\times X}$ and $(Y, p_*\mathbf{T}_X) \times (Y, p_*\mathbf{T}_X)$ are identical.

Finally, we note that the preceding reasoning applies to the quotient group projection $\pi : G \to G/H$ because π is an open mapping.■

6. We claim that the quotient group is the multiplicative group $S^1 \subset \mathbb{C}$ of complex numbers such that $|z| = 1$. Consider the map $p : \mathbb{R} \to S^1$ defined by $p(t) = (\cos 2\pi\, t, \sin 2\pi\, t)$. Standard trigonometric identities imply that $p(t_1 + t_2) = p(t_1) \cdot p(t_2)$ for all $t_1, t_2 \in \mathbb{R}$. This map is continuous and onto, and the kernel is $\mathbb{Z}$. Therefore p passes to a continuous and 1–1 onto homomorphism $p^* : \mathbb{R}/\mathbb{Z} \to S^1$. Furthermore, the quotient group is compact because the restriction of p to the compact subset $[0, 1]$ is onto. Therefore Theorem III.1.9 implies that p^* is a homeomorphism, and this completes the proof that p^* is an isomorphism of topological groups.■

7. (a) If U is an open neighborhood of 1, then by continuity of multiplication there are open neighborhoods W_1 and W_2 of 1 such that $W_1 \cdot W_2 \subset U$. If

$$V \;=\; (W_1 \cap W_2) \;\cap\; (W_1 \cap W_2)^{-1}$$

then the identity $g = \left(g^{-1}\right)^{-1}$ implies that $V = V^{-1}$, and we also have $V \cdot V \subset W_1 \cdot W_2 \subset U$.■

(b) [Recall that Munkres' definition implies that G is $\mathbf{T}_1$.]

Let $U = G - \{y^{-1}x\}$, so that U is an open neighborhood of 1, and take V as in (a). We claim that $x \cdot V$ and $y \cdot V$ are disjoint. Assume that they do have some point z in common. Then we have $z = xv_1 = yv_2$ for some $v_1, v_2 \in V$, and thus we also have $y^{-1}x = v_2 \cdot v_1^{-1}$. Since V is symmetric and $V \cdot V \subset U$, it follows that $y^{-1}x \in U$, contradicting our choice of U. The source of the contradiction was our assumption regarding the existence of z, and therefore no such point can exist; in other words, we must have $(x \cdot V(\cap(y \cdot V) = \emptyset$.∎

(c) Let $U = G - (x^{-1} \cdot A)$, so that $1 \in U$ and U is disjoint from the closed set $B = x^{-1} \cdot A$. Once again, take V as in (a). We claim that the open subsets $x \cdot V$ and $A \cdot V$ are disjoint. If not, then we have $xv_1 = av_2$ for some $V - 1, v_2 \in V$ and $a \in A$. We can rewrite the equation in the form $x^{-1}a = v_2 \cdot v_1^{-1}$, and as in (b) we have $x^{-1}a \in U$, which contradicts our choice of V. The source of the contradiction was the assumption that $x \cdot V$ and $A \cdot V$ had a point in common, and therefore these sets must be disjoint.∎

(d) Follow the hint. Suppose that E is closed in G/H and $\pi(x) \notin E$. Taking inverse images, we see that $xH = \pi^{-1}[\pi[x]]$ is disjoint from the set $F = \pi^{-1}[E]$, which is closed and has the property that $F \times H = F$ (*i.e.*, F is right H-invariant). The reasoning of (c) implies that there is a symmetric neighborhood V of 1 such that $V \cdot x$ and $V \cdot F$ are disjoint, and the same also holds for $V \cdot xH$ and $V \cdot F$ because $F \cdot H = H$. Since V is open it follows that $V \cdot C$ is open for all $C \subset G$, and therefore $V \cdot xH$ and $V \cdot F = V \cdot F \cdot H$ are also open.

Consider the open sets $\pi[V \cdot xH]$ and $\pi[V \cdot F]$ in G/H. The first of these contains $\pi(x)$, and the second contains F. Since the respective inverse images of these open sets are the disjoint subsets $V \cdot xH$ and $V \cdot F = V \cdot F \cdot H$, it follows that $\pi[V \cdot xH]$ and $\pi[V \cdot F]$ are disjoint open subsets containing $\pi(x)$ and F respectively. Since each of the latter was arbitrary, this implies that G/H is regular.∎

Problems from Munkres, § 26, *pp.* 170 – 172

12. See Theorem 1 in `proper.pdf`; the first half of the proof shows that, more generally, if a closed continuous (not necessarily onto) map satisfies the hypotheses, then inverse images of compact subsets are compact; although the statement of the theorem is restricted to Hausdorff spaces, the proof does not use this assumption on X and Y.∎

13. [The definition of topological group in Munkres includes an assumption that one point subsets are closed. Although this additional hypothesis is not universally adopted, we need it in order to solve the exercise. By Exercise 22.7 in Munkres, the assumption implies that G is also Hausdorff and even satisfies the $\mathbf{T}_3$ separation axiom.]

(a) We shall follow the hint, so suppose that $c \notin A \cdot B$ where $A \subset G$ is closed and $B \subset G$ is compact. The for all $b \in B$ we have $c \notin A \cdot b$, and therefore there are two disjoint open sets U_b and V_b such that $c \in U_b$ and $A \cdot b \subset V_b$.

We claim there is an open neighborhood W_b of b such that $A \cdot b \subset W_b$. — By continuity of multiplication, for each $a \in A$ there is an open neighborhood N_a of b such that $ab \in a \cdot N_a \subset V_b$. If W_b is the union of all the neighborhoods N_a, then $A \cdot W_b \subset V_b$. The open sets W_b define an open covering of B, so by compactness there is a finite subcovering $W_{b_1}, \cdots, W_{b_k}$. If $U_c = U_{b_1} \cap \cdots \cap U_{b_k}$, then $c \in U$ and $U \cap A \cdot B = \emptyset$. Therefore $G - (A \cdot B)$ contains an open neighborhood of c for each $c \notin A \cdot B$, and consequently $G - (A \cdot B)$ is open, or equivalently $A \cdot B$ is closed in G.∎

(b) Suppose that $F \subset G$ is closed. Then by (a) we know that $E \cdot H$ is also closed. Since p is onto, we have

$$p^{-1}[p[F]] \quad = \quad F \cdot H$$

and since p is a quotient map this implies that $p[F]$ is closed in G/H.■

(c) The projection $p : G \to G/H$ is closed by (b), and the inverse image of each point $[g] \in G/H$ is the coset gH, which is homeomorphic to the compact subgroup H. Therefore by Exercise 26.12 (proved above) we know that G is compact.■

Problems from Munkres, § 30*, pp.* 194 – 195

18. We shall follow the hint. Let $\mathcal{B} == \{B_n\}$ be a countable neighborhood base at $1 \in G$. Taking intersections $B_n \cap B_n^{-1}$ if necessary, we might as well assume that the open sets B_n are symmetric (with respect to taking inverses). Furthermore, by passing to a subsequence if necessary, we might as well assume that $B_{n+1} \cdot B_{n+1} \subset B_n$ (this uses the first part of Exercise 22.7 and the fact that we have a neighborhood base; taken together, these imply that there is some $k > n$ such that $B_k \cdot B_k \subset B_n$).

The basic idea is to imitate a proof that the additive groups $\mathbb{R}^n$ are second countable under the given hypotheses. The countable neighborhood base should serve as a weak substitute for the $\frac{1}{2^n}$-neighborhoods centered at points. — This underlying principle turns out to be very useful in working with topological groups; for example, one can use neighborhood bases at the identity to formulate a strong analog of uniform continuity for continuous maps of topological groups.

Suppose first that G has a countable dense subset D, and let U be an open subset of G. For each $x \in U$ the set $x^{-1} \cdot U$ is an open neighborhood of 1 and hence there is some open set $B_n \in \mathcal{B}$ such that $B_n \subset x^{-1} \cdot U$. The latter implies $x \cdot B_n \subset U$. Let $d \in D$ be such that $d \in x \cdot B_{n+1}$; then $x^{-1}d \in B_{n+1}$, and since B_n is symmetric we also have $d^{-1}x \in B_{n+1}$, which in turn implies that $x \in d \cdot B_{n+1}$. We now have

$$d \cdot B_{n+1} \quad = \quad x \cdot x^{-1} d \cdot B_{n+1} \quad \subset \quad x \cdot B_{n+1} \cdot B_{n+1} \subset x \cdot B_n$$

which means that U is a union of open sets in the countable family $\{ \ d \cdot B_k \mid d \in D, k \in \mathbb{N}^* \ \}$, and therefore the latter is a countable base for G.

We shall now assume that G is Lindelöf. For each $B_k \in \mathcal{B}$, consider the open covering $\{ \ g \cdot B_k \mid g \in G \ \}$ It has a finite subcovering, so choose $g_{k,j}$ such that the sets $g_{k,j} \cdot B_k$ form a countable subcovering, and let D be the set of points $g_{k,j}$ as k and j both run through all positive integers. This is a countable family, and we claim it is dense in G. As before, let U be an open subset of G and let $x \in G$, so that there is an open neighborhood of x having the form $x \cdot B_n \subset U$ for some n. By construction there is an open set $d \cdot B_{n+1}$ containing x, and as in the preceding discussion we have $d \in d \cdot B_{n+1} \subset x \cdot B_n$. Since the latter is contained in U, it follows that $d \in U$, which implies that D is dense in G. We can now apply the reasoning of the first part of the problem to conclude that G is second countable.■

Problems from Munkres, § 31*, pp.* 199 – 200

8. Let $p : X \to X/G$ denote the orbit space projection map (which is continuous since we are taking the quotient topology on X/G.)

There are several parts to this exercise, with one for each of the listed properties. It seems best to begin with some preliminary observations as in Section II.3 of the book by Bredon which is cited below.

CLAIM 1. *The orbit space projection $X \to X/G$ is an open mapping, with no compactness hypothesis on G.* — If U is open in X, then $G \cdot U = \cup_g \, g \cdot U$ is also open, and thus $p^{-1}[p[U]] = G \cdot U$ shows that $p[U]$ is open in X/G.

CLAIM 2. *If G is compact, then the orbit space projection $X \to X/G$ is closed.* — This follows by the same sort of reasoning used in Exercise 26.13(i). Let $F \subset X$ be closed. If we can show that $G \cdot F$ is closed when G is compact, then $p^{-1}[p[F]] = G \cdot F$ will show that $p[F]$ is closed in X/G. To show that $G \times F$ is closed, suppose that $x \notin G \cdot F$. Then for all $g \in G$ we have $x \notin G \cdot F$. For each $g \in G$ there is an open neighborhood V_g of $g \cdot x$ such that $V_g \cap A = \emptyset$ By continuity there is a neighborhood U_g of g such that $U_g \cdot V_g \subset X - A$. If $\mathcal{U}$ is the open covering of G by the sets U_g, then by compactness there is a finite subcovering $U_{g(1)}, \cdots, U_{g(k)}$. The corresponding open subsets $V_{g(j)}$ intersect in some open neighborhood V_x of x, and we have $G \cdot V_x \subset U_j \cdot V_j \subset X - A$. The latter implies that V_x is an open neighborhood of x which is contained in $X - G \cdot A$. Since $x \in X - G \cdot A$ was arbitrary, it follows that $X - G \cdot A$ is open in X and hence $G \cdot A$ is closed in X.■

CLAIM 3. *If G is compact, then the orbit space projection $X \to X/G$ is proper: Inverse images of compact subsets are compact.* — This follows from Exercise 1 in `proper.pdf` or equivalently from Exercise 26.12.

With these at our disposal, we can prove the statements in the exercise fairly directly.

Hausdorff. Suppose $[x] \neq [y]$ in X/G. If we lift everything back to X, this translates into a statement that the orbits $G \cdot x$ and $G \cdot y$ are disjoint. Each orbit is compact because it is a continuous image of G, so we have a pair of disjoint compact subsets. We can now proceed as in Section VI.3 to find disjoint open neighborhoods U_0 and V_0 of $G \cdot x$ and $G \cdot y$. By Wallace's Theorem there are open neighborhoods U and V of x and y such that $G \cdot U \subset \alpha^{-1}[U_0]$ and $G \cdot V \subset \alpha^{-1}[V_0]$. It follows that $p[G \cdot U]$ and $p[G \cdot V]$ are disjoint open neighborhoods of $[x]$ and $[y]$.■

Regularity. Let $[x] \in X/G$, and let $C \subset X/G$ be a closed subset such that $[x] \notin C$. Then $G \cdot x$ is a compact subset of X which is disjoint from the G-invariant closed subset $F = p^{-1}[C]$ (invariance means $G \cdot F = F$). In an arbitrary regular space Y, standard arguments show that if K is a compact subset which is disjoint from the closed subset E, then there are disjoint open subsets U and V containing K and E (this is left to the reader). If we specialize this to the case where $K = G \cdot x$ and $E = F$, then as in the preceding argument there is some open neighborhood W of x such that $G \times W \subset U$. It then follows that $[x]$ and C have disjoint open neighborhoods given by $p[G \cdot W]$ and $p[V]$.■

Normality. Suppose that A and B are disjoint closed subsets of X/G, and let E and F denote the inverse images, which are G-invariant disjoint closed subsets of X. Since X is normal, there are disjoint open subsets U and V containing E and F, and as before there are open neighborhoods M and N of U and V such that $G \times M \subset U$ and $G \times N \subset V$. It follows that $p[G \times M]$ and $p[G \times N]$ are disjoint open subsets containing A and B.■

Local compactness. Let $x \in X$, and suppose that x has an open neighborhood U such that $U \subset C$ where C is compact. Then $p[U]$ is an open neighborhood of $[x] = p(x)$ and $p[U]$ is contained in the compact subset $p[C]$.■

Second countability. Let $\mathcal{B}$ be a countable base for X, and define a family of open subsets in X/G by taking $V_k = p[B_k]$ for $B_k \in \mathcal{B}$. We claim that the sets V_k form a countable base for the topology on X/G. Suppose that W is open in X/G, and let $W' = p^{-1}[W]$. Then there is some subsequence of positive integers $k(n)$ such that $W' = \cup_n \, B_{k(n)}$, and since $p[W'] = W$ we have $W = \cup_n \, p[B_{k(n)}] = \cup_n \, V_{k(n)}$.■

10. This proof is very similar to the proof of Urysohn's Lemma. Since the course notes only give a reference for the proof of Urysohn's Lemma, we shall follow suit and note that the solution to this exercise is on pages 29–30 of the book by Montgomery and Zippin (which is listed in the bibliography at the end of the course notes).

Note. Several other topological properties of orbit space projections (including some relevant to the second part of this course) are given in Chapter I and Section II.6 of the following book:

G. E. Bredon. Introduction to Compact Transformation Groups. Pure and Applied Mathematics Series Vol. 46. *Academic Press, New York*, 1972.

We should note that the conjecture stated at the bottom of page 38 in that book has been shown to be true. One proof is given in Section IV.14 of another book by the same author which is cited below, and some of the result's implications for more general orbit spaces are summarized in Section 1 of the paper by Donnelly and Schultz.

G. E. Bredon. Sheaf Theory (Second Edition). Graduate Texts in Mathematics Vol. 170. *Springer-Verlag, New York etc.*, 1997.

H. Donnelly and R. Schultz. Compact group actions and maps into aspherical manifolds. *Topology* **21** (1982), 443–455.

Additional exercises

Notation. Let $\mathbb{F}$ be the real or complex numbers. Within the matrix group $\mathbf{GL}(n, \mathbb{F})$ there are certain subgroups of particular importance. One such subgroup is the *special linear group* $\mathbf{SL}(n, \mathbb{F})$ of all matrices of determinant 1.

0. We shall follow the steps indicated in the hint.

If N is a normal subgroup of $\mathbf{SL}(2, \mathbb{C})$ and $A \in N$ then N contains all matrices that are similar to A.

If B is similar to A then $B = P\,A\,P^{-1}$ for some $P \in \mathbf{GL}(2, \mathbb{C})$, so the point is to show that one can choose P so that $\det P = 1$. The easiest way to do this is to use a matrix of the form βP for some nonzero complex number β, and if we choose the latter so that $\beta^2 = (\det P)^{-1}$ then βP will be a matrix in in $\mathbf{SL}(2, \mathbb{C})$ and B will be equal to $(\beta P)\,A\,(\beta P)^{-1}$.■

Therefore the proof reduces to considering normal subgroups containing a Jordan form matrix of one of the following two types:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \quad \begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$$

Here α is a complex number not equal to 0 or ± 1 and $\varepsilon = \pm 1$.

The preceding step that if N contains a given matrix then it contains all matrices similar to that matrix. Therefore if N is a nontrivial normal subgroup of $\mathbf{SL}(2, \mathbb{C})$ then N is a union of similarity classes, and the latter in turn implies that N contains a given matrix A if and only if it contains a Jordan form for A. For 2×2 matrices there are only two basic Jordan forms; namely, diagonal matrices and elementary 2×2 Jordan matrices of the form

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where $\lambda \neq 0$. Such matrices have determinant 1 if and only if the product of the diagonal entries is the identity, and this means that the Jordan forms must satisfy conditions very close in the hint; the only difference involves the diagonal case where the diagonal entries may be equal to ± 1. If N is neither trivial nor equal to the subgroup $\{\pm I\}$, then it must contain a Jordan form given by a diagonal matrix whose nonzero entries are not ± 1 or else it must contain an elementary 2×2 Jordan matrix.

The idea is to show that if N contains one of these Jordan forms then it contains all such forms, and this is done by computing sufficiently many matrix products. Trial and error is a good way to approach this aspect of the problem.

The basic strategy is to show first that if N contains either of the matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \quad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

(where $\alpha \neq 0, \ \pm 1$), then N also contains

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and next to show that if N contains the latter matrix then N contains every Jordan form. As noted before, the latter will imply that $N = \mathbf{SL}(2, \mathbb{C})$.

At this point one needs to do some explicit computations to find products of matrices in N with sufficiently many Jordan forms. Suppose first that N contains the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} .$$

Then N contains A^2, which is equal to

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and the latter is similar to

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

so N contains B. Suppose now that N contains some diagonal matrix A whose nonzero entries are α and α^{-1} where $\alpha \neq \pm 1$, and let B be given as above. Then N also contains the commutator $C = A\,B\,A^{-1}\,B^{-1}$. Direct computation shows that the latter matrix is given as follows:

$$C = \begin{pmatrix} 1 & \alpha^2 - 1 \\ 0 & 1 \end{pmatrix}$$

Since $\alpha \neq \pm 1$ it follows that C is similar to B and therefore $B \in N$.

We have now shown that if N is a nontrivial normal subgroup that is not equal to $\{\pm I\}$, then N must contain B. As noted before, the final step is to prove that if $B \in N$ then $N = \mathbf{SL}(2, \mathbb{C})$.

Since $B \in N$ implies that N contains all matrices similar to B it follows that all matrices of the forms

$$P = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad , \qquad Q = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

(where z is an arbitrary complex number)

also belong to N, which in turn shows that $PQ \in N$. But.

$$PQ = \begin{pmatrix} 1 & z \\ z & z^2 + 1 \end{pmatrix}$$

and its characteristic polynomial is $t^2 - (z^2 + 2)\,t - 1$. If $z \neq 0$ then this polynomial has distinct nonzero roots such that one is the reciprocal of the other. Furthermore, if $\alpha \neq 0$ then one can always solve the equation $\alpha + \alpha^{-1} = z^2 + 2$ for z over the complex numbers and therefore we see that for each $\alpha \neq 0$ there is a matrix PQ similar to

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

such that $PQ \in N$. It follows that each of the displayed diagonal matrices also lies in N; in particular, this includes the case where $\alpha = -1$ so that the diagonal matrix is equal to $-I$. To complete the argument we need to show that N also contains a matrix similar to

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} .$$

But this is relatively easy because N must contain $-B = (-I)B$ and the latter is similar to A.■

FOOTNOTE. More generally, if $\mathbb{F}$ is an arbitrary field then every proper normal subgroup of $\mathbf{SL}(n, \mathbb{F})$ is contained in the central subgroup of diagonal matrices of determinant 1; of course, the latter is isomorphic to the group of n^{th} roots of unity in $\mathbb{F}$.

Definition. The *orthogonal group* $\mathbf{O}(n)$ consists of all transformations in $\mathbf{GL}(n, \mathbb{R})$ that take each orthonormal basis for $\mathbb{R}^n$ to another orthonormal basis, or equivalently, the subgroup of all matrices whose columns form an orthonormal basis. It is an easy exercise in linear algebra to show that the determinant of all matrices in $\mathbf{O}(n)$ is ± 1. The *special orthogonal group* $\mathbf{SO}(n)$ is the subgroup of $\mathbf{O}(n)$ consisting of all matrices whose determinants are equal to $+1$. Replacing If we replace the real numbers $\mathbb{R}$ by the complex numbers $\mathbb{C}$ we get the unitary groups $\mathbf{U}(n)$ and the *special unitary groups* $\mathbf{SU}(n)$, which are the subgroups of $\mathbf{U}(n)$ given by matrices with determinant 1. The determinant of every matrix in $\mathbf{U}(n)$ is of absolute value 1 just as before, but in the complex case this means that the determinant is a complex number on the unit circle. In Appendix A the orthogonal and unitary groups were shown to be compact.

1. The group $\mathbf{GL}(n, \mathbb{R})$ is isomorphic to the multiplicative group of nonzero real numbers. Therefore $\mathbf{O}(1)$ is isomorphic to set of nonzero real numbers that take 1 to an element of absolute value 1; but a nonzero real number has this property if and only if its absolute value is 1, or equivalently if it is equal to ± 1.

Consider now the group $\mathbf{SO}(2)$. Since it sends the standard basis into an orthonormal basis, it follows that its columns must be orthonormal. Therefore there is some θ such that the entries of the first column are $\cos\theta$ and $\sin\theta$, and there is some φ such that the entries of the second column are $\cos\varphi$ and $\sin\varphi$. Since the vectors in question are perpendicular, it follows that $|\theta-\varphi| = \pi/2$, and depending upon whether the sign of $\theta-\varphi$ is positive or negative there are two possibilities:

(1) If $\theta - \phi > 0$ then $\cos\varphi = -\sin\theta$ and $\sin\varphi = \cos\theta$.

(2) If $\theta - \phi < 0$ then $\cos\varphi = \sin\theta$ and $\sin\varphi = -\cos\theta$.

In the first case the determinant is 1 and in the second it is -1.

We may now construct the isomorphism from S^1 to $\mathbf{SO}(2)$ as follows: If $z = x + y\,i$ is a complex number such that $|z| = 1$, send z to the matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} .$$

This map is clearly 1–1 because one can retrieve z directly from the entries of the matrix. It is onto because the complex number $\cos\theta + i\,\sin\theta$ maps to the matrix associated to θ. Verification that the map takes complex products to matrix products is an exercise in bookkeeping.

A homeomorphism from $\mathbf{SO}(2)\times\mathbf{O}(1)$ to $\mathbf{O}(2)$ may be constructed as follows: Let $\alpha : \mathbf{O}(1) \to \mathbf{O}(2)$ be the map that sends ± 1 to the diagonal matrix whose entries are ± 1 and 1, and define $M(A,\varepsilon)$ to be the matrix product $A\,\alpha(\varepsilon)$. Since the image of -1 does not lie in $\mathbf{SO}(2)$, standard results on cosets in group theory imply that the map M is 1–1 and onto. But it is also continuous, and since it maps a compact space to a Hausdorff space it is a homeomorphism onto its image, which is $\mathbf{SO}(2)$.

To see that this map is not a group isomorphism, note that in the direct product group $\mathbf{SO}(2)\times\mathbf{O}(1)$ there are only finitely many elements of order 2 (specifically, the first coordinate must be $\pm I$). On the other hand, the results from Appendix D show that all elements of $\mathbf{O}(2)$ that are not in $\mathbf{SO}(2)$ have order 2 and hence there are infinitely many such element in $\mathbf{O}(2)$. Therefore the latter cannot be isomorphic to the direct product group.■

2. This is similar to the first part of the preceding exercise. The group $\mathbf{GL}(n,\mathbb{C})$ is isomorphic to the multiplicative group of nonzero complex numbers. Therefore $\mathbf{U}(1)$ is isomorphic to set of nonzero complex numbers that take 1 to an element of absolute value 1; but a nonzero complex number has this property if and only if its absolute value is 1, or equivalently if it lies on the circle S^1.■

3. The proof separates into three cases depending upon whether the group in question is $\mathbf{U}(n)$, $\mathbf{SU}(n)$ or $\mathbf{SO}(n)$,

The case $G = \mathbf{U}(n)$.

We start with the unitary group. The Spectral Theorem states that if A is an $n\times n$ unitary matrix then there is another invertible unitary matrix P such that $B = P\,A\,P^{-1}$ is diagonal. We claim that there is a continuous curve joining B to the identity. To see this, write the diagonal entries of B as $\exp it_j$ where t_j is real and $1 \le j \le n$. Let $C(s)$ be the continuous curve in the diagonal unitary matrices such that the diagonal entries of $C(s)$ are $\exp(i\,st_j)$ where $s\in[0,1]$. It follows immediately that $C(0) = I$ and $C(1) = B$. Finally, if we let $\gamma(s) = P^{-1}\,C(s)\,P$ then γ is a continuous curve in $\mathbf{U}(n)$ such that $\gamma(0) = I$ and $\gamma(1) = A$. This shows that $\mathbf{U}(n)$ is in fact arcwise connected.■

The case $G = \mathbf{SU}(n)$.

The preceding argument also shows that $\mathbf{SU}(n)$ is arcwise connected; it is only necessary to check that if A has determinant 1 then everything else in the construction also has this property. The determinants of B and A are equal because the determinants of similar matrices are equal. Furthermore, since the determinant of B is 1 it follows that $\sum_j t_j = 0$, and the latter in turn implies that the image of the curve $C(s)$ is contained in $\mathbf{SU}(n)$. Since the latter is a normal subgroup of $\mathbf{U}(n)$ it follows that the curve $\gamma(s)$ also lies in $\mathbf{SU}(n)$, and therefore we conclude that the latter group is also arcwise connected.

The product decomposition for $\mathbf{U}(n)$ is derived by an argument similar to the previous argument for $\mathbf{O}(2)$. More generally, suppose we have a group G with a normal subgroup K and a second subgroup H such that $G = H \cdot K$ and $H \cap K = \{1\}$. Then group theoretic considerations yield a 1–1 onto map $\varphi : K \times H \to G$ given by $\varphi(k, h) = k \cdot h$. If G is a topological group and the subgroups have the subspace topologies then the 1–1 onto map φ is continuous. Furthermore, if G is compact Hausdorff and H and K are closed subgroups of G then φ is a homeomorphism. In our particular situation we can take $G = \mathbf{U}(n)$, $K = \mathbf{SU}(n)$ and $H \cong \mathbf{U}(1)$ to be the subgroup of diagonal matrices with ones down the diagonal except perhaps for the $(1,1)$ entry. These subgroups satisfy all the conditions we have imposed and therefore we have that G is homeomorphic to the product $K \times H$.■

The case $G = \mathbf{SO}(n)$.

Finally, we need to verify that $\mathbf{SO}(n)$ is arcwise connected, and as indicated in the hint we use the normal form obtained in Appendix D. According to this result, for every orthogonal $n \times n$ matrix A there is another orthogonal matrix P such that $B = P\,A\,P^{-1}$ is is a block sum of orthogonal matrices that are either 2×2 or 1×1. These matrices may be sorted further using their determinants (recall that the determinant of an orthogonal matrix is ± 1).

In fact, one can choose the matrix P such that the block summands are sorted by size and determinant such that

(1) the 1×1 summands with determinant 1 come first,
(2) the 2×2 summands with determinant 1 come second,
(3) the 2×2 summands with determinant -1 come third,
(4) the 1×1 summands with determinant -1 come last.

Each matrix of the first type is a rotation matrix R_θ where the first row has entries $(\cos\theta \quad -\sin\theta)$, and each matrix of the second type is a matrix S_θ where the first row has entries $(\cos\theta \quad \sin\theta)$.

The first objective is to show that B lies in the same arc component as a matrix with no summands of the second type. Express the matrix B explicitly as a block sum as follows:

$$B = I_k \oplus \left(\bigoplus_{i=1}^{p} R_{\theta(i)} \right) \oplus \left(\bigoplus_{j=1}^{q} R_{\varphi(j)} \right) \oplus -I_\ell$$

Here I_m denotes an $m \times m$ identity matrix. Consider the continuous curve in $\mathbf{SO}(n)$ defined by the formula

$$C_1(t) = I_k \oplus \left(\bigoplus_{i=1}^{p} R_{t\theta(i)} \right) \oplus I_{2q+\ell} \ .$$

It follows that $C_1(1) = B$ and $C_1(0) = B_1$ is a block sum matrix with no summands of the second type.

The next objective is to show that B_1 lies in the same arc component as a matrix with no summands of the third type. This requires more work than the previous construction, and the initial step is to show that one can find a matrix in the same arc component with at most one summand of the third type. The crucial idea is to show that if the block sum has $q \geq 2$ summands of the third type, then it lies in the same arc component as a matrix with $q-2$ block summands of the third type. In fact, the continuous curve joining the two matrices will itself be a block sum, with one 4×4 summand corresponding to the two summands that are removed and identity matrices corresponding to the remaining summands. Therefore the argument reduces to looking at a 4×4 orthogonal matrix that is a block sum of two 2×2 matrices of the third type, and the objective is to show that such a matrix lies in the same arc component as the identity matrix.

Given real numbers α and β, let S_α and S_β be defined as above, and likewise for R_α and R_β. Then one has the multiplicative identity

$$S_\alpha \oplus S_\beta \quad = \quad \big(R_\alpha \oplus R_\beta\big) \cdot \big(1 \oplus (-1) \oplus 1 \oplus (-1)\big) \ .$$

We have already shown that there is a continuous curve $C(t)$ in the orthogonal group such that $C(0) = I$ and $C(1) = R_\alpha \oplus R_\beta$. If we can construct a continuous curve $Q(t)$ such that $Q(0) = I$ and $Q(1) = \big(1 \oplus (-1) \oplus 1 \oplus (-1)\big)$. Then the matrix product $C(t)\,Q(t)$ will be a continuous curve in the orthogonal group whose value at 0 is the identity and whose value at 1 is $S_\alpha \oplus S_\beta$. But $R_\alpha \oplus R_\beta$ is a rotation by 180 degrees in the second and fourth coordinates and the identity on the first and third coordinates, so it is natural to look for a curve $Q(t)$ that is rotation through $180\,t$ degrees in the even coordinates and the identity on the odd ones. One can write down such a curve explicitly as follows:

$$Q(t) \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \pi t & 0 & -\sin \pi t \\ 0 & 0 & 1 & 0 \\ 0 & \sin \pi t & 0 & \cos \pi t \end{pmatrix}$$

Repeated use of this construction yields a matrix with no block summands of the first type and at most one summand of the second type. If there are no summands of the third type, we have reached the second objective, so assume that there is exactly one summand of the third type. Since our block sum has determinant 1, it follows that there is also at least one summand of the fourth type. We claim that the simplified matrix obtained thus far lies in the same arc component as a block sum matrix with no summands of the third type and one fewer summands of the fourth type. As in the previous step, everything reduces to showing that a 3×3 block sum $S_\alpha \oplus (-1)$ lies in the same arc component as the identity. In this case one has the multiplicative identity

$$S_\alpha \oplus (-1) \quad = \quad \big(R_\alpha \oplus 1\big) \oplus \Big(1 \oplus (-1) \oplus (-1)\Big)$$

and the required continuous curve is given by the formula

$$C(t) \quad = \quad \big(R_{t\alpha} \oplus 1\big) \oplus \Big((1 \oplus R_{t\pi}(-1)\Big) \ .$$

Thus far we have shown that every block sum matrix in $\mathbf{SO}(n)$ lies in the same arc component as a diagonal matrix (only summands of the first and last types). Let $I_s \oplus -I_\ell$ be this matrix. Since its determinant is 1, it follows that ℓ must be even, so write $\ell = 2k$. Then the diagonal matrix is a block sum of an identity matrix and k copies of R_π, and one can use the continuous curve $D(t)$

that is a block sum of an identity matrix with k copies of $R_{t\pi}$ to show that the diagonal matrix $I_s \oplus -I_\ell$ lies in the same arc component as the identity.

We have now shown that every block sum matrix with determinant 1 lies in the arc component of the identity. Suppose that $\Gamma(t)$ is a continuous curve whose value at 0 is the identity and whose value at 1 is the block sum. At the beginning of this proof we noted that for every orthogonal matrix A we can find another orthogonal matrix P such that $B = P\,A\,P^{-1}$ is is a block sum of the type discussed above; note that the determinant of B is equal to 1 if the same is true for the determinant of A. If $\Gamma(t)$ is the curve joining the identity to B, then $P^{-1}\,\Gamma(t)\,P$ will be a continuous curve in $\mathbf{SO}(n)$ joining the identity to A.■

4. One can use the same sort of arguments that established topological product decompositions for $\mathbf{O}(2)$ and $\mathbf{SU}(n)$ to show that $\mathbf{O}(n)$ is homeomorphic to $\mathbf{SO}(n) \times \mathbf{O}(1)$. Since $\mathbf{SO}(n)$ is connected, this proves that $\mathbf{O}(n)$ is homeomorphic to a disjoint union of two copies of $\mathbf{SO}(n)$.■

5. Let A be an invertible $n \times n$ matrix over the real or complex numbers, and consider the meaning of the Gram-Schmidt process for matrices. Let $\mathbf{a}_j$ represent the j^{th} column of A, and let B be the orthogonal or unitary matrix whose columns are the vectors $\mathbf{b}_j$ obtained from the columns of A by the Gram-Schmidt process. How are they related? The basic equations for defining the columns of B in terms of the columns of A have the form

$$\mathbf{b}_j = \sum_{k \leq j} c_{k,j}\mathbf{a}_k$$

where each $c_{j,j}$ is a positive real number, and if we set $c_{k,j} = 0$ for $k > j$ this means that $B = AC$ where C is lower triangular with diagonal entries that are positive and real. We claim that there is a continuous curve $\Gamma(t)$ such that $\Gamma(0) = I$ and $\Gamma(1) = C$. Specifically, define this curve by the following formula:

$$\Gamma(t) \quad = \quad I \; + \; t\,(C - I)$$

By construction $\Gamma(t)$ is a lower triangular matrix whose diagonal entries are positive real numbers, and therefore this matrix is invertible. If we let $\Phi(t)$ be the matrix product $A\,\Gamma(t)$ then we have a continuous curve in the group of invertible matrices such that $\Phi(1) = A$ and $\Phi(0)$ is orthogonal or unitary. Therefore it follows that every invertible matrix is in the same arc component as an orthogonal or unitary matrix.

In the unitary case this proves the result, for the arcwise connectedness of $\mathbf{U}(n)$ and the previous argument imply that $\mathbf{GL}(n, \mathbb{C})$ is also arcwise connected. However, in the orthogonal case a little more work is needed. The determinant of a matrix in $\mathbf{GL}(n, \mathbb{R})$ is either positive or negative, so the preceding argument shows that an invertible real matrix lies in the same arc component as the matrices of $\mathbf{SO}(n)$ if its determinant is positive and in the other arc component of $\mathbf{SO}(n)$ if its determinant is negative. Therefore there are at most two arc components of $\mathbf{GL}(n, \mathbb{R})$, and the assertion about arc components will be true if we can show that $\mathbf{GL}(n, \mathbb{R})$ is not connected. To see this, note that the determinant function defines a continuous onto map from $\mathbf{GL}(n, \mathbb{R})$ to the disconnected space $\mathbb{R} - \{0\}$.■

6. We need to show that the open neighborhood $\mathcal{N} = \mathcal{N}(\Phi)$ of $\{1\} \times X$ (the points where Φ is defined) is equal to all of $G \times X$. By Property (1) of a local group action, we know that $\{1\} \times X \subset \mathcal{N}$. Since X is compact, by Wallace's Theorem (see Section VI.2) we know that there is some open neighborhood V of 1 in G such that $V \times X \subset \mathcal{N}$. If we replace V by $U = V \cap V^{-1}$, then $U \times X \subset \mathcal{N}$ and $U = U^{-1}$ is an open neighborhood of the identity.

Since U is connected and $U = U^{-1}$, we know that every open neighborhood of U generates G. Given $g \in G$, write

$$g \;=\; \prod_{j=1}^{m} u_j^{\varepsilon_j}$$

where $u_j \in U$ and $\varepsilon_j = \pm 1$. If p_j is the product of the last j factors (so $p_1 = u_m^{\varepsilon_m}$ and $p_j = u_{m+1-j}^{\varepsilon_{m+1-j}} \cdot p_{j-1}$ recursively), then induction on j and Property (2) imply that $(p_j, x) \in \mathcal{N}$ for $j = 1, \cdots, m$. Since $p_m = g$, it follows that for all x and all g we have $(g, x) \in \mathcal{N}$ and hence $\mathcal{N} = G \times X$.∎