SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 7

Fall 2014

VII. Topological deformations and approximations

VII.0 : Categories and functors

Additional exercises

1. (a) Follow the hint and prove the contrapositives.

Not a monomorphism $\Rightarrow$ *not* $1-1$. If f is not a monomorphism, then there exist mappings $g, h : C \to A$ such that $g \neq h$ but $f \circ g = f \circ h$. Now $g \neq h$ means that there is some $x \in C$ such that $g(x) \neq h(x)$, and by hypothesis we have $f(g(x)) = f \circ g(x) = f \circ h(x) = f(h(x))$, so that f sends both $g(x)$ and $h(x)$ to the same element of B. But this means that f is not 1–1.■

Not an epimorphism $\Rightarrow$ *not onto.* If f is not an epimorphism, then there exist mappings $u, v : B \to D$ such that $u \neq v$ but $u \circ f = v \circ f$. The functional identity translates to the identity $u(f(a)) = v(f(a))$ for all $a \in A$, and therefore we have $u|f[A] = v|f[A]$. On the other hand, $u \neq v$ implies that $u(b) \neq v(b)$ for some $b \in B$, and by the previous sentence we know that b cannot belong to $f[A]$. Therefore $f[A]$ is a proper subset of B, which means that f is not onto.■

(b) Suppose that $f[A]$ is dense in B and $u \circ f = u \circ g$, where $u, v : B \to D$. Then u and v are equal on the dense subset $f[A]$. Since D is Hausdorff, the set E of all points b such that $u(b) = v(b)$ is closed. We know that E contains the dense subset $f[A]$, so we must have $E = B$.■

> *Note.* The *Wikipedia* article `http://en.wikipedia.org/wiki/Epimorphism` gives extensive information on the relationship between epimorphisms and surjective mappings for many standard examples of categories. Frequently, but not always, these notions are equivalent.

(c) Suppose we are given $f_1 : A \to B$ and $f_2 : B \to C$.

Assume both maps are monomorphisms. Let g and h be morphisms into A such that $(f_2 \circ f_1) \circ g = (f_2 \circ f_1) \circ h$. By associativity of composition and the monomorphism hypothesis on f_2, we have $f_1 \circ g = f_1 \circ h$; but now the monomorphism hypothesis on f_1 implies that $g = h$.■

Assume both maps are epimorphisms. Let u and v be morphisms from C such that $u \circ (f_2 \circ f_1) = v \circ (f_2 \circ f_1)$. By associativity of composition and the epimorphism hypothesis on f_1, we have $u \circ f_2 = v \circ f_2$; but now the epimorphism hypothesis on f_2 implies that $u = v$.■

(d) Suppose that f and g are morphisms $W \to X$ such that $r \circ f = f \circ g$, and let q be such that $q \circ r = \mathrm{id}_X$. Then we have

$$f \;=\; \mathrm{id}_X \circ f \;=\; q \circ r \circ f \;=\; q \circ r \circ g \;=\; \mathrm{id}_X \circ g \;=\; g$$

which means that r is a monomorphism.

(e) Suppose that u and v are morphisms $B \to D$ such that $u \circ p = v \circ p$. Then we have

$$u \;=\; u \circ \mathrm{id}_B \;=\; u \circ p \circ s \;=\; v \circ p \circ s \;=\; v \circ \mathrm{id}_B \;=\; v$$

which means that p is a epimorphism.

> *Note.* In the category of sets, the Axiom of Choice implies that every monomorphism is a retract and every epimorphism is a retraction, but for other categories this fails. For example, in the category of abelian groups the monomorphism $\mathbb{Z}_2 \to \mathbb{Z}_4$ is not a retract and the epimorphism $\mathbb{Z}_4 \to \mathbb{Z}_2$ is not a retraction. [PROOF: In the first case, if there was a homomorphism $q : \mathbb{Z}_4 \to \mathbb{Z}_2$ such that $q|\mathbb{Z}_2$ was the identity, then the fact that the image of $\mathbb{Z}_2$ is $2\mathbb{Z}_4$ implies that $q|2\mathbb{Z}_4$ is nonzero, and this cannot happen since $2\mathbb{Z}_2 = 0$. In the second case, if one could find a suitable homomorphism s then the image of s would be contained in $2\mathbb{Z}_4$ and once again this would yield the contradictory conclusion $p \circ s = 0$.]

2. We shall first dispose of the converse. Assume f is an isomorphism, and let $g : B \to A$ be its inverse. Then by associativity of composition we know that $\mathrm{Mor}(f, \cdots) \circ \mathrm{Mor}(g, \cdots)$ sends a morphism $h : A \to X$ to $h \circ g \circ f = h \circ \mathrm{id}_A = h$, so that $\mathrm{Mor}(f, \cdots) \circ \mathrm{Mor}(g, \cdots)$ is the identity on $\mathrm{Mor}(A, X)$. Similarly, by associativity of composition we know that $\mathrm{Mor}(g, \cdots) \circ \mathrm{Mor}(f, \cdots)$ sends a morphism $k : B \to X$ to $h \circ f \circ g = k \circ \mathrm{id}_B = k$, so that $\mathrm{Mor}(g, \cdots) \circ \mathrm{Mor}(f, \cdots)$ is the identity on $\mathrm{Mor}(B, X)$.

We shall now prove the exercise itself. Assume that $\mathrm{Mor}(f, \cdots)$ is an isomorphism from $\mathrm{Mor}(B, X)$ to $\mathrm{Mor}(A, X)$ for all X. If we let $X = A$ this means that there is a unique $p : B \to A$ such that $p \circ f = \mathrm{id}_A$. To prove the result we want, it suffices to show that $f \circ p = \mathrm{id}_B$. Since $\mathrm{Mor}(f, \cdots)$ is 1–1, it will suffice to prove that $(f \circ p) \circ f = \mathrm{id}_B \circ f = f$; but this follows immediately from $p \circ f = \mathrm{id}_A$.■

3. (*a*) For the empty set, the unique map is the one whose graph is the empty subset of $\emptyset \times A = \emptyset$ (this function is often called the *empty map*). For the one point set, the unique map is the one whose graph is the entire set $A \times \{\text{point}\}$ (in other words, the only possible constant map into the set).■

(*b*) Since a linear transformation always sends a zero vector to a zero vector, there is only one possibility for the map if the vector space consists only of the zero vector, so a zero space is an initial object. On the other hand, for a linear transformation into a zero space there is only possible value for the transformaton at a point; namely, the zero vector. Therefore a zero space is also a terminal object.■

(*c*) We shall first consider initial objects. Since there is an identity map from an initial object $\mathcal{O}$ to itself and by hypothesis there is only one self-map of $\mathcal{O}$, it follows that a map from $\mathcal{O}$ to itself must be the identity. Suppose now that $\mathcal{O}$ and $\mathcal{O}'$ are initial objects. Then there are unique maps $a : \mathcal{O} \to \mathcal{O}'$ and $b : \mathcal{O}' \to \mathcal{O}$. By the uniqueness of self-maps for initial objects, it follows that $b \circ a$ is the identity for $\mathcal{O}$ and $a \circ b$ is the identity for $\mathcal{O}'$. Therefore a and b are both isomorphisms.

We next consider terminal objects. Since there is an identity map from a terminal object $\mathcal{T}$ to itself and by hypothesis there is only one self-map of $\mathcal{T}$, it follows that a map from $\mathcal{T}$ to itself must be the identity. Suppose now that $\mathcal{T}$ and $\mathcal{T}'$ are terminal objects. Then there are unique maps $a : \mathcal{T} \to \mathcal{T}'$ and $b : \mathcal{T}' \to \mathcal{T}$. By the uniqueness of self-maps for terminal objects, it follows that $b \circ a$ is the identity for $\mathcal{T}$ and $a \circ b$ is the identity for $\mathcal{T}'$. Therefore a and b are both isomorphisms.■

4. If Z is a null object and A is an arbitrary object, then there are unique maps $A \to Z$ and $Z \to A$, so for each null object we obtain a unique composite morphism $A \to A$. We need to show that if W is any other null object, then the composites $A \to Z \to A$ and $A \to W \to A$ are equal. For the sake of definiteness, let $t_Z : A \to Z$ and $j_Z : Z \to A$ be the unique maps from and to the null object Z, and similarly for W. Now let $a : W \to Z$ and $b : Z \to W$ be the unique maps between the two null objects; then $a \circ b$ and $b \circ a$ must be identity maps by uniqueness, and consequently a and b are isomorphisms which are inverse to each other. Therefore we have

$$j_W \circ t_W \quad = \quad j_W \circ \mathrm{id}_W \circ t_W \quad = \quad j_W \circ b \circ a \circ t_W$$

and since $a \circ t_W = t_Z$ and $j_W \circ b = j_Z$ by uniqueness, it follows that $j_W \circ t_W = j_Z \circ t_Z$, and hence the composite does not depend upon a the choice of a specific null object.■

4. Suppose that $r : X \to Y$ is a retract and $s : Y \to X$ is a map such that $q \circ r = \mathrm{id}_X$. If F is a covariant functor defined on the category under consideration then we have

$$\mathrm{id}_{F(X)} \quad = \quad F(\mathrm{id}_X) \quad = \quad F(q \circ r) \quad = \quad F(q) \circ F(r)$$

and consequently $F(r)$ is also a retract. Similarly, if $p : A \to B$ is a retraction and $p \circ s = \mathrm{id}_B$, then we have

$$\mathrm{id}_{F(B)} \quad = \quad F(\mathrm{id}_B) \quad = \quad F(p \circ s) \quad = \quad F(p) \circ F(s)$$

and consequently $F(p)$ is also a retraction.

If G is a contravariant functor the conclusions are more complicated; namely, if r is a retract then $G(r)$ is a retraction, and if p is a retraction then $G(p)$ is a retract. The derivations are nearly the same as the preceding ones, but at the last step one must reverse the orders of composition.■

5. In fact, f is a retract, for there is a unique morphism $c : X \to E$ because E is terminal, and as noted before the morphism $c \circ f : E \to E$ must be the identity if E is terminal.■

6. This is basically a translation of fundamental statements about matrix multiplication into the language of category theory. A morphism $A : n \to m$ is merely an $m \times n$ matrix over the integers, the identity matrix is the identity morphism, and composition is matrix multiplication. The composition rules for identities and associativity are then merely restatements of the corresponding properties of matrix multiplication.■

7. Let $f : X \to Y$ be a morphism, and suppose that f has a quasi-inverse $g : Y \to X$; we claim there is some $h : Y \to X$ such that f is a quasi-inverse to h. The natural first candidate is $h = g$, but this does not lead anywhere so we need to find another choice for h. The correct choice is $g \circ f \circ g$, and the string of equations

$$(gfg)f(gfg) \quad = \quad (gfg)(fgf)g \quad = \quad (gfg)fg \quad = \quad g(fgf)g \quad = \quad gfg$$

shows that f is a quasi-inverse to $g \circ f \circ g$ (the composition operators were omitted in the display to make the equations easier to follow).■

> *Note.* Usually morphisms in a category do not have quasi-inverses, but in the category of sets the Axiom of Choice is essentially equivalent to the existence of quasi-inverses.

VII.1 : Homotopic mappings

Problems from Munkres, § 51*, p.* 330

2. On the unit interval $I = [0, 1]$ the identity map is homotopic to a constant map by convexity. Therefore we have the following:

(a) If $f : X \to I$ is continuous, then $f = \mathrm{id}_I \circ f$ is homotopic to $C_0 \circ f$, where C_0 is the constant function whose value everywhere is zero. Since $C_0 \circ f = C_0$, it follows that f is homotopic to the map which sends everything to zero.■

(b) If $f : I \to Y$ is continuous, then $f = f \circ \mathrm{id}_I$ is homotopic to $f \circ C_0$, which is the constant map with value $f(0)$ everywhere. Since Y is arcwise connected, one can use a curve joining $f(0)$ to an arbitrary point $y_0 \in Y$, and hence all constant maps are homotopic to each other. Combining these, we see that every continuous mapping $I \to Y$ is homotopic to the constant map with value y_0.■

3. (a) More generally, if $K \subset \mathbb{R}^n$, then the straight line homotopy $H(x,t) = (1-t)x + tx_0$ is a homotopy from the identity on K to the constant map whose value everywhere is x_0.■

(b) If $H : X \times [0,1] \to X$ is the homotopy from the identity to the constant map with value x_0, then $H|\{x\} \times [0,1]$ is a curve joining x to x_0, and therefore every point in X lies in the arc component of x_0.■

(c) This is essentially the same argument as in 2(a) with I replaced by an arbitrary contractible space.■

(d) This is essentially the same argument as in 2(b) with I replaced by an arbitrary contractible space.■

Problem from Munkres, § 52, *p.* 334

1. (a) If $A \subset \mathbb{R}^2$ is the union of the $x-$ and $y-$axes, then A is star convex with a_0 taken to be the origin because $x \in A$ and $0 < t < 1$ implies $t\,x \in A$. However, A is not convex because $(1,0)$ and $(0,1)$ are in A but $\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(1,0) + \frac{1}{2}(0,1)$ does not lie in A.■

(b) [*Not included because the concept of simple connectedness has not yet been introduced. However, another exercise has a stronger result; namely, a star convex set is contractible.*]

Additional exercises

1. The set of continuous maps $P \to X$ is in 1–1 correspondence with the points of X such that $f : P \to X$ corresponds to $f(p_0) \in X$. Two such continuous mappings are homotopic if and only if their values $f_0(p_0)$ and $f_1(p_0)$ can be joined by a continuous curve in X. Therefore one obtains a 1–1 correspondence between $[I, X]$ and the set of arc components of X by sending f to the arc component of $f(p_0)$.■

2. The idea is to follow the hint and prove that two of maps from X to Y are always homotopic. As noted in the hint, if Y has the indiscrete topology and W is an arbitrary topological space, then every map of sets from W to Y is continuous. In particular, if $V \subset W$ and $g : V \to Y$ is continuous, then there is an extension of g to a map of sets from W to Y, and this extension is automoatically continuous. In particular, this is true if $W = X \times [0,1]$ and $V = X \times \{0,1\}$, proving that if f_0 and f_1 are continuous mappings from X to the indiscrete space Y, then one can always construct a homotopy from f_0 to f_1.■

3. Since a continuous map takes connected sets to connected sets and the connected components of a discrete space are the one point subsets, it follows that every continuous map $X \to Y$ is a constant map and, in addition, every homotopy between two continuous maps is also constant. Therefore, if we take the 1–1 correspondence between points of Y and constant maps from X to Y, we obtain a map $Y \to [X, Y]$ which is both 1–1 and onto.■

4. (i) Star convexity implies that the image of the straight line homotopy $H(x,t) = (1-t)\,a_0 + t\,x$ is contained in A, so it follows that the identity on A is homotopic to the constant map whose value is a_0.■

(ii) Follow the hint. First of all, if K and L are convex and $p \in K \cap L$, then $K \cup L$ is star convex with respect to p because $x \in K \cup L$ implies $x \in K$ or $x \in L$, and in either case the line segment defined by $(1-t)\,p + t\,x$ will be contained in $K \cup L$. In the example, K and L are convex subsets of $\mathbb{R}^2$, and one can check that $(7/4, 3/4)$ and $(3/4, 7/4)$ are in $A = K \cup L$. However, their midpoint is $(5/4, 5/4)$, and this point is not in A.■

VII.2 : Some examples

Additional exercises

1. Let $x \in U$, and choose $\delta > 0$ such that $N_\delta(x)$ is contained in U. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, it follows that there is some point $y \in N_\delta(x) \cap \mathbb{Q}^n$. Now $N_\delta(x)$ is arcwise connected and hence is contained in the arc component of x, so we have shown that this arc component contains a point in $\mathbb{Q}^n$. It follows that we can define a 1–1 map from the arc components of U to $\mathbb{Q}^n$ by choosing some point in $\mathbb{Q}^n$ for each arc component. Since $\mathbb{Q}^n$ is countable, this means the set of arc components must also be countable.■

2. Let $H : X \times [0,1] \to Y$ be the homotopy from f_0 to f_1, and consider the image B of $A \times [0,1]$ where A is an arcwise connected subset of X. Since $A \times [0,1]$ is arcwise connected, it follows that B is arcwise connected and hence is contained in an arc component of Y. The conclusion of the exercise now follows because both $f_0[A]$ and $f_1[A]$ are contained in B.■

VII.3 : Homotopy classes of mappings

Problem from Munkres, § 58, pp. 366 – 367

6. The first thing to notice is that the arcwise connectedness of X implies that all constant maps into X are homotopic, and consequently if the identity on X is homotopic to a constant map, it is also homotopic to a constant map whose value lies in the subspace A. Now let $i : A \to X$ be a retract, with $r : X \to A$ such that $r \circ i = \mathrm{id}_A$, and let $H : X \times [0,1] \to X$ be a homotopy from the identity to a constant map whose value lies in A. Then the composite

$$h'(a,t) \quad = \quad r \circ H(a,t)$$

is a homotopy from the identity on A to a constant map.

Problems from Hatcher, pp. 18 – 20

4. Let $j : A \to X$ be the inclusion map, and let $g : X \to A$ be the map $g(x) = f_1(x)$, which exists because the image of f_1 is contained in A. By construction we then know that $j \circ g = f_1$ is homotopic to f_0, which is the identity on X. To prove that $g \circ j$ is homotopic to the identity on A, proceed as follows: Since each map f_t maps A into itself, it follows that f_t induces a homotopy

$H : A \times [0,1] \to A$ such that $H(a,t) = f_t(a)$ for all a and t. It then follows that H defines a homotopy from the identity on A to $f_1|A = j \circ g$.∎

10. We shall first prove the statement in the first sentence. If X is contractible, then the identity map on X is homotopic to a constant, so if $f : X \to Y$ is arbitrary then $f = f \circ \mathrm{id}_X$ is homotopic to $f \circ C_0$ for some constant map C_0; since the latter composite is also a constant map, this shows that f must be nullhomotopic. Conversely, the hypothesis $[X,Y] = \{\text{point}\}$ for all Y specializes to the case $Y = X$ and the statement that the homotopy class of the identity map in $[X,X]$ must be equal to the homotopy class of the constant map.∎

12. If $f : X \to Y$ is a homotopy equivalence and P is a one point set, then $f_* : [P,X] \to [P,Y]$ must be an isomorphism; since $[P,Z]$ is the set of arc components of Z by a previous exercise, it follows that f induces a 1–1 correspondence between the arc components of X and the arc components of Y.

The proof of the corresponding statement for connected components is also elementary but slightly different. We can express things formally as follows: If $g : A \to B$ is a continuous map, then for each connected component C of A we know that the connected set $g[A]$ must lie in a connected component of B, and therefore if $\mathbf{ConnComp}(E)$ denotes the set of connected components of a space E, then the continuous mapping g induces a map of sets

$$\mathbf{ConnComp}(g) : \mathbf{ConnComp}(A) \longrightarrow \mathbf{ConnComp}(B)$$

and it is an elementary exercise to verify that this defines a covariant functor on the category of topological spaces and continuous maps. Furthermore, since arcwise connected sets are connected, a variant on Additional Exercise VII.2.2 implies that if g_0 and g_1 are homotopic maps from A to B, then the induced maps of connected components satisfy the homotopy invariance property

$$\mathbf{ConnComp}(g_0) \quad = \quad \mathbf{ConnComp}(g_1) \; .$$

Combining these observations, we see that if f is a homotopy equivalence from X to Y then the map $\mathbf{ConnComp}(f)$ will be an isomorphism.

To prove the statement in the final sentence, observe that we have the following commutative diagram, in which we identify $[P,Z]$ with the arc components of a space Z and the horizontal maps send an arc component of a space Z to the connected component which contains it.

$$\begin{array}{ccc}
[P,X] & \xrightarrow{f_*} & [P,Y] \\
\downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\
\mathbf{ConnComp}(X) & \xrightarrow{\mathbf{ConnComp}(f)} & \mathbf{ConnComp}(Y)
\end{array}$$

If f is a homotopy equivalence, then the results in the first two paragraphs imply that f_* and $\mathbf{ConnComp}(f)$ are isomorphisms. Therefore if either of the maps α_X or α_Y is an isomorphism, then so is the other.∎

Additional exercises

1. The key point to observe is that f and g are homotopy inverses to f^{-1} and g^{-1} respectively. Therefore it follows that $\mathrm{id}_X = f^{-1} \circ f \simeq f^{-1} \circ g$, yielding a relationship chain

$$g^{-1} \quad = \quad \mathrm{id}_X \circ g^{-1} \quad = \quad f^{-1} \circ g \circ g^{-1} \quad = \quad f{-}1 \circ \mathrm{id}_X \quad = \quad f^{-1}$$

which shows that $f^{-1} \simeq g^{-1}$.■

2. (i) Follow the hints. For the first part, suppose that i is a retract. If $r : X \to A$ is such that $r \circ i = \mathrm{id}_A$, then by functoriality $r_* \circ i_* = (i = \mathrm{id}_A)_*$, and since the latter is the identity on $[A, Y]$ it follows that r_* is a retract, and consequently r_* is 1–1. For the second part, let $Y = A = \{0, 1\} \subset [0, 1] = X$ as in the hints. Since A is a discrete space, every self-map of A is continuous, and no two self-maps are homotopic (use a previous exercise), so the set $[A, A]$ contains exactly four elements. On the other hand, since X is contractible the set $[A, X]$ contains exactly one element by a previous exercise. Therefore the map $[A, A] \to [X, A]$ cannot be 1–1.■

(ii) In this case we have $i^* \circ r^* = (i = \mathrm{id}_A)^*$, and since the latter is the identity on $[Z, A]$ it follows that i^* is a retraction, and consequently i^* is onto. To show that i^* need not be onto, take $Z = A$ and X as in the first part of the exercise. Then $[A, X]$ consists of one point and $[A, A]$ consists of four points, and therefore $i^* : [A, X] \to [A, A]$ is not onto for this example.■

3. (i) In the setting of this exercise, a map $f : Z \to X \times Y$ is totally determined by $p \circ f$ and $q \circ f$; two maps are equal if and only if their coordinate projections are equal, and if we are given $Z \to X$ and $Z \to Y$ then this pair arises as the coordinate projections of a continuous map $Z \to X \times Y$. Similarly, if H is a homotopy from f to f', then $p \circ H$ and $q \circ H$ define homotopies of their respective coordinate projections, and if we have homotopies of the coordinate projections then they arise as the coordinate projections of some homotopy $Z \times [0, 1] \to Y$. These observations combine to prove that coordinate projections define an isomorphism θ_Z.

To prove the naturality property of θ_Z (*i.e.*, the commuative diagram), note that if $g : W \to Z$ is continuous, then the coordiante projections of $f \circ g$ are merely $p \circ f \circ g$ and $q \circ f \circ g$ respectively.■

(ii) Given homotopy classes $u, v, w \in [Y, X]$, choose representative continuous functions r, s, t respectively, and let C_1 denote the constant map $Y \to X$ whose value is always 1. The associative law on homotopy classes $u \cdot (v \cdot w) = (u \cdot v) \cdot W$ follows directly from the associativity identity

$$[r \cdot (s \cdot t)](y) \quad = \quad r(y)\, s(y)\, t(y) \quad = \quad [(r \cdot s) \cdot t](y)$$

which comes from the associativity of m and holds for all $y \in Y$. Similarly, $u \cdot [C_1] = u = [C_1] \cdot u$ follows from $r(y) \cdot 1 = r(y) = 1 \cdot r(y)$. Finally, an inverse to the homotopy class of u is given by the function $q(y) = r(y)^{-1}$, for we have $q \cdot r = C_1 = r \cdot q$ by the same reasoning as above.

If $h : Y \to Z$ is continuous, then h^* sends u, v, w to classes represented by $r \circ h$, $s \circ h$ and $t \circ h$. Therefore $h^*(u \cdot v)$ is represented by $(r \cdot s) \circ h$, which is equal to $(r \circ h) \cdot (s \circ h)$. Since the latter represents $h^*(u) \cdot h^*(v)$, it follows that h^* defines a group homomorphism with respect to the group structure on $[Y, X]$.

> *Note.* If $X = S^1$, then this group structure is abelian because S^1 is abelian, and the resulting abelian group — often called the *Bruschlinsky group* — is used in the file `polishcircle.pdf`.

4. (a) The straight line homotopy H_t from id_A to the constant map with value a_0 is a basepoint preserving homotopy. Therefore if $f : (A, a_0) \to (X, x_0)$ is a basepoint preserving map, then $f \circ H_t$ is a basepoint preserving homotopy from f to the basepoint preserving constant map.■

(b) Let H_t be as in part (a), and suppose we are given a basepoint preserving map $g : (X, x_0) \to (A, a_0)$. Then $H_t \circ g$ is a basepoint preserving homotopy from f to the basepoint preserving constant map.■

Note. In contrast to some previous exercises, there was no need to assume X was arcwise connected in (a) because a basepoint preserving map will always send the arcwise connected space A into the arc component of the basepoint in X.

5. (i) As suggested in the hint, let g be a homotopy inverse to f. Then the associated maps of homotopy classes $(f \circ g)_* = f_* \circ g_*$, $(g \circ f)_* = g_* \circ f_*$, $(f \circ g)^* = g^* \circ f^*$, and $(g \circ f)^* = f^* \circ g^*$ are all identity maps because $f \circ g$ and $g \circ f$ are homotopic to identity maps. It follows that g_* is an inverse to f_* and g^* is an inverse to f^*.

(ii) This was already done in Exercise 12 on page 19 of Hatcher, for which the solution was given above.∎

6. The Cantor set X is given as the intersection of closed subsets X_n, where each X_n is a finite union of pairwise disjoint intervals, each of which has length 3^{-n}. If $u, v \in X$ lie in the same arc component, then for each n they must lie on one of these intervals and hence the distance between them is at most 3^{-n}, so that $|u - v| < 3^{-n}$ for all n. But this can happen only if $u = v$. Therefore the arc components of X are one point sets, and there are countably many of them. If X were homotopy equivalent to an open subset of some $\mathbb{R}^n$, then by an earlier exercise the open set and X would have to contain only countably many components. Consequently, there is no homotopy equivalence from X to an open subset of some $\mathbb{R}^n$.∎

VII.4 : Homotopy types

Problem from Munkres, § 58*, pp.* 366 – 367

1. Note that "deformation retract" in Munkres means "strong deformation retract" in the sense of this course. We now proceed to the proof of the exercise.

Let $i : A \to B$ and $j : B \to X$ be the inclusions, and let $p : B \to A$ and $q : X \to B$ be the maps given by the deformation retract data. Then $(p \circ q) \circ (j \circ i) = p \circ (q \circ j) \circ i = p \circ \mathrm{id}_B \circ i = \mathrm{id}_A$, so it only remains to show that $(j \circ i) \circ (p \circ q)$ is homotopic to the identity relative to A. We know that $i \circ p$ is homotopic to the identity relative to A, and since $pq|A$ is the identity it follows that $(j \circ i) \circ (p \circ q)$ $(j \circ (i \circ p) \circ q)$ is homotopic to $(j \circ \mathrm{id}_A \circ q) = j \circ q$ relative to A. Since the right hand side is homotopic to the id_X relative to B and $A \subset B$, it follows that $(j \circ i) \circ (p \circ q)$ is homotopic to the identity relative to A. This means that A is a strong deformation retract of X.∎

Problems from Hatcher, pp. 18 – 20

5. Let $H : X \times [0, 1] \to X$ be a homotopy from the identity to the constant map C_x, and let W be the open set $H^{-1}[W]$. This open subset contains $\{x\} \times [0, 1]$, so by Wallace's Theorem there is an open neighborhood V of x such that $V \times [0, 1] \subset W$. It follows that $H|V \times [0, 1]$ defines a homotopy into U from the inclusion $V \subset U$ to the constant map C_x on V.∎

13. To simplify the notation, we shall denote the deformation retract data by $H : X \times [0, 1] \to X$ and $K : X \times [0, 1] \to X$ respectively. It will also be convenient to denote the boundary of $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$ (the four edge segments) by Γ.

By the definition of deformation retract data, the homotopies H and K satisfy the following conditions:

(1) If $a \in A$, then $H(a,t) = K(a,t) = a$ for all $t \in [0,1]$.

(2) We have $H(x,0) = K(x,0) = x$ for all $x \in X$.

(3) We have $H(x,1) \in A$ and $K(x,1) \in A$ for all $x \in X$.

The goal of the exercise is to construct a homotopy from H to K; in other words, we want a map $L : X \times [0,1] \times [0,1] \to X$ with the following additional properties:

(4) We have $L(X,0,t) = H(x,t)$ and $L(X,1,t) = K(x,t)$ for all $x \in X$ and $t \in [0,1]$.

(5) If $a \in A$, then $L(a,s,t) = a$ for all $s,t \in [0,1]$.

(6) We have $L(x,s,1) \in A$ for all $x \in X$ and $s \in [0,1]$.

(7) We have $L(x,s,0) = x$ for all $x \in X$ and $s \in [0,1]$.

If L exists, then we can view the maps $L_s = L|X \times \{s\} \times [0,1]$ as a 1-parameter family of deformation retract data starting with H and ending with K.

The first step in constructing L is to define $D : X \times [0,1] \times [0,1] \to X$ by the formula

$$D(x,s,t) \quad = \quad H\,(\,K(x,s),\,t\,)\ .$$

The hypotheses on H and K imply that D has the following properties:

(8) We have $D(X,s,0) = K(x,s)$ and $D(X,0,t) = H(x,t)$ for all $x \in X$ and $s,t \in [0,1]$.

(9) If $a \in A$, then $L(a,t,s) = K(a,t,s) = a$ for all $s,t \in [0,1]$.

(A) We have $L(x,s,t) \in A$ and for all $x \in X$ if either $s = 1$ or $t = 1$.

(B) We have $D(x,0,0) = x$ for all $x \in X$.

Evidently the behavior of $D|X \times \Gamma$ does not fit the requirements for L that we have listed, but the properties of this restriction suggest that we can realize the requirements for L if we compose D with $\mathrm{id}_X \times \theta$, where $\theta : [0,1] \times [0,1] \to [0,1] \times [0,1]$ is a continuous map with the following behavior on Γ:

(C) The bottom edge $[0,1] \times \{0\}$ is collapsed to $(0,0)$.

(D) The left edge $\{0\} \times [0,1]$ is mapped to itself by the identity.

(E) The top edge $[0,1] \times \{1\}$ maps to the union of the top and right edges $[0,1] \times \{1\} \cup \{1\} \times [0,1]$ such that $[0, \frac{1}{2}] \times \{1\}$ maps to the top edge such that $(t,1)$ is sent to $(2t,1)$ and $[\frac{1}{2},1] \times \{1\}$ maps to the right edge such that $(t,1)$ is sent to $(1, 2-2t)$.

(F) The right edge $\{1\} \times [0,1]$ maps to the bottom edge $[0,1] \times \{0\}$ such that $(1,t)$ is sent to $(t,0)$.

The drawing in `math205Aexercises7a.pdf` illustrates the behavior we want on Γ, and a suitable function θ is constructed in that document. One can then use the definitions to verify that the function $L(x;s,t) = D(x;\theta(s,t))$ has all the desired properties.■

Additional exercises

1. In words, the subset A consists of the bottom edge and the side edges of the rectangle X. As usual, we shall follow the approach in the hint; there is a drawing for this exercise in `math205Aexercises7a.pdf`.

The retraction $r : [-1,1] \times [0,1] \to [-1,1] \times \{0\} \ \cup \ \{-1,1\} \times [0,1]$ is defined by a radial projection with center $(\mathbf{0},2) \in [-1,1] \times \mathbb{R}$. As indicated by the drawing, the formula for r depends upon whether $2|x| + t \geq 2$ or $2|x| + t \leq 2$. Specifically, if $2|x| + t \geq 2$ then

$$r(x,\, t) \quad = \quad \frac{1}{|x|}\left(x,\, 2|x| + t - 2\right)$$

while if $2|x| + t \leq 2$ then we have

$$r(x,\, t) \quad = \quad \frac{1}{2}\left((2-t)x,\, 0\right)$$

and these are consistent when $2|x|+t = 2$ then both formulas yield the value $|x|^{-1}(x,\, 0)$. Elementary but slightly tedious calculation then implies that $r(x,\, t)$ always lies in $[-1,1] \times [0,1]$, and likewise that r is the identity on $[-1,1] \times \{0\} \ \cup \ \{-1,1\} \times [0,1]$. The homotopy from inclusion$\circ r$ to the identity is then the straight line homotopy

$$H(x,\, t;\, s) \quad = \quad (1-s)\cdot r(x,\, t) \ + \ s \cdot (x,\, t)$$

and this completes the proof of the exercise.■

2. Follow the hint: Let $X' \subset X \times [0,1]$ be the subset $X \times \{0\} \cup A \times [0,1]$, and let $A' = A \times \{1\}$.

If $h : X \to X'$ is the composite $X \cong X \times \{0\} \subset X'$, let $\rho : X' \to X$ maps $X \times \{0\}$ to X by projection onto the first coordinate and maps $A \times [0,1]$ to A similarly. Then Exercise 4 implies that $X \times \{0\}$ is a deformation retract of X' and hence h is a homotopy equivalence. Take the mapping h_0 to be the slice embedding $A \cong A \times \{1\}$. A homotopy from $h|A$ to $j \circ h_0$ is given by $K(a,t) = (a,t)$. Finally, we need to find an open neighborhood U of A' in X' such that A' is a strong deformation retract of U, and this can be done by taking $U = A \times (0,1]$. Since $A' = A \times \{1\}$ it follows that A' is a strong deformation retract of U, so the only thing remaining is to prove that U is open in X'. We can see this most easily by viewing everything as a subspace of $X \times [0,1]$; in particular, since $V = X \times (0,1]$ is open in $X \times [0,1]$ and $X' \cap V = A \times (0,1] = U$, it follows that U is open in X'.■

3. For each α we have mappings $g_\alpha : Y_\alpha \to X_\alpha$ such that $g_\alpha \circ f_\alpha$ is homotopic to the identity on X_α and $f_\alpha \circ g_\alpha$ is homotopic to the identity on Y_α; denote these homotopies by H_α and K_α respectively.

We claim that $\prod_\alpha \ g_\alpha$ is a homotopy inverse to $\prod_\alpha \ f_\alpha$, and this requires the construction of homotopies

$$\Phi : \left(\prod_{\alpha \in A} \ X_\alpha\right) \times [0,1] \ \longrightarrow \ \prod_{\alpha \in A} \ X_\alpha \ , \qquad \Psi : \left(\prod_{\alpha \in A} \ Y_\alpha\right) \times [0,1] \ \longrightarrow \ \prod_{\alpha \in A} \ Y_\alpha$$

such that the composites

$$\prod_{\alpha \in A} \ g_\alpha \circ f_\alpha \quad = \quad \left(\prod_{\alpha \in A} \ g_\alpha\right) \ \circ \ \left(\prod_{\alpha \in A} \ f_\alpha\right)$$

$$\prod_{\alpha \in A} \ f_\alpha \circ g_\alpha \quad = \quad \left(\prod_{\alpha \in A} \ f_\alpha\right) \ \circ \ \left(\prod_{\alpha \in A} \ g_\alpha\right)$$

are respectively homoptopic to the identity maps on $\prod_{\alpha\in A}\ X_\alpha$ and $\prod_{\alpha\in A}\ Y_\alpha$.

As usual, it suffices to define the coordinate projections of Φ and Ψ for each $\gamma \in A$, and we do so by the formulas

$$\pi_\gamma{}^\circ\Phi = H_\gamma{}^\circ\left(\pi_\gamma \times \mathrm{i}f_{[0,1]}\right)\ , \qquad \pi_\gamma{}^\circ\Psi = K_\gamma{}^\circ\left(\pi_\gamma \times \mathrm{i}f_{[0,1]}\right)\ .$$

It follows immediately that these maps define homotopies from

$$\left(\prod_{\alpha\in A}\ g_\alpha\right)\ ^\circ\ \left(\prod_{\alpha\in A}\ f_\alpha\right) \qquad \text{and} \qquad \left(\prod_{\alpha\in A}\ f_\alpha\right)\ ^\circ\ \left(\prod_{\alpha\in A}\ g_\alpha\right)$$

to the identity mappings.■

4. (i) Let $r : F \to B$ and $h : F \times [0,1] \to F$ be the deformation retraction data which exist because B is a strong deformation retract of F. Extend r to $r' : X \to A$ by letting $r'|A$ be the identity, and extend h to $H : X \times [0,1] \to X$ by letting $H|A \times [0,1]$ be projection onto the first factor. If these maps are continuous, then they for deformation retract data for $A \subset X$. But the continuity of these extensions follows because $F \cap A = B$, and the restriction of r to B is the identity, and the restrictions of h to $B \times [0,1]$ is projection onto the first factor.■

(ii) For $i = 1, 2$ let r_i and $H^{(i)}$ be deformation retract data for $C \subset F_i$. Then the restrictions of these data to X and $X \times [0,1]$ are the identity and projection onto the first factor respectively, so we can assemble r_1 and r_2 into a continuous mapping $r : X \to C$, and likewise we can assemble $H^{(1)}$ and $H^{(2)}$ into a continuous mapping $H : X \times [0,1] \to X$. The properties of the deformation retract data for $C \subset F_i$ imply that r and H are deformation retract data for $C \subset X$.■