# Drawing for Hatcher, Exercise 13, page 19

This drawing is meant to motivate and clarify a crucial step in our solution for the cited exercise. We need to find a continuous mapping from the solid square $[0, 1] \times [0, 1]$ to itself with the following behavior on the edges:

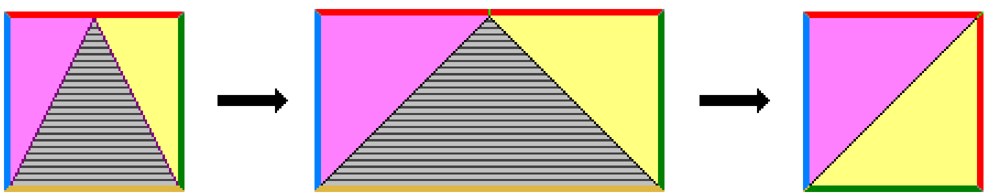
(1) The bottom edge (in ***gold***) is collapsed to the corner point $(0, 0)$.

(2) The left edge (in ***blue***) is mapped to itself.

(3) The top edge (in ***red***) maps to the union of the top and right edges.

(4) The right edge (in ***green***) maps to the bottom edge.

The drawing and discussion below suggest one way of constructing such a function.



It is convenient to break the mapping into a twofold composite, where the first step is to stretch the square $[0, 1] \times [0, 1]$ to the rectangle $[0, 2] \times [0, 1]$ in the obvious fashion. The second step is to collapse the rectangle back into the square, such that the (pink) triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$ is mapped to itself by the identity, the (yellow) triangle with vertices $(1, 1)$, $(2, 1)$, and $(2, 0)$ is mapped to the congruent triangle with vertices $(1, 1)$, $(1, 0)$, and $(0, 0)$ such that $(1, 1)$ corresponds to itself, $(2, 1)$ corresponds to $(1, 0)$, and $(2, 0)$ corresponds to $(0, 0)$; also, the (striped gray) triangle with vertices $(0, 0)$, $(1, 1)$, and $(2, 0)$ is collapsed along horizontal lines onto the closed segment joining the first two vertices. Explicit formulas defining this mapping are given as follows; to simplify the notation we shall refer to the triangles by their colors (we have already defined them formally):

(1) On the pink triangle, which consists of the points $(u, v)$ in the rectangle such that $u \leq v$, the map sends $(u, v)$ to itself.

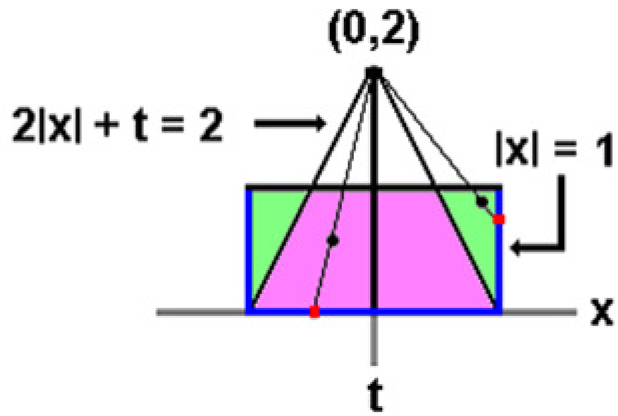
(2) On the striped gray triangle, which consists of the points $(u, v)$ in the rectangle such that $v \leq u \leq 2 - v$, the map sends $(u, v)$ to $(v, v)$.

(3) On the yellow triangle, which consists of the points $(u, v)$ in the rectangle such that $2 - v \leq u$, the map sends $(u, v)$ to $(v, 2 - u)$.

The first two formulas are easy to derive, and the third can be done using linear algebra in various ways. For example, this can be done by finding the unique affine transformation of the coordinate plane which takes the points $(1, 1)$, $(2, 1)$, and $(2, 0)$ to the points $(1, 1)$, $(1, 0)$, and $(0, 0)$ respectively, or one can derive the formula by observing that the map on the yellow triangle is a clockwise rotation through $90$ degrees which is centered at $(1, 1)$. Either way, the formula for the mapping turns out to be the one which is given in the third case. One can check directly that these definitions yield the same values for points on more than one triangle, so the formulas yield a well defined function.

# Drawing to accompany Additional Exercise VII.4.1

Here is a picture of the retraction described in the hint:




In this illustration, the retraction from $[-1,1] \times [0,1]$ to $[-1,1] \times \{0\} \cup \{-1,1\} \times [0,1]$ (the blue line segments) sends the points marked in black into the points marked in red on the corresponding lines. The explicit definition of the retraction has two cases, depending upon whether or not the original point lie in the pink colored region or the green colored region(s).

A similar argument shows that $\mathbf{D}^n \times [0,1] \cup \mathbf{S}^{n-1} \times [0,1]$ is a retract of $\mathbf{D}^n \times [0,1]$ for all positive integers $n$. For example, one can obtain the case $n = 2$ from the $1-$dimensional case by taking solids and surfaces of revolution about the $t-$axis, and likewise in higher dimensions one can view the drawing as a planar cross section of the general construction.