SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 8

Fall 2014

VIII. The fundamental group

VIII.1 : Definitions and basic properties

Problems from Munkres, § 52, *pp.* 334 − 335

2. We shall start by translating the conclusion into the notation in the course notes. Given a curve θ, Munkres' notation $\widehat{\theta}$ refers to the curve we call $-\theta$; for a curve defined on $[0,1]$ we have $-\theta(t) = \theta(1-t)$. So in our terminology the identity to be verified is

$$-(\alpha + \beta) = (-\beta) + (-\alpha) .$$

Recall that $\{\alpha+\beta\}(t) = \alpha(2t)$ if $t \le \frac{1}{2}$ and $\{\alpha+\beta\}(t) = \beta(2t-1)$ if $t \le \frac{1}{2}$. Therefore $-\{\alpha+\beta\}(t) = \{\alpha+\beta\}(1-t)$ is given by

(a) $\alpha(2(1-t)) = \alpha(2-2t)$ if $(1-t) \le \frac{1}{2}$ or equivalently if $t \ge \frac{1}{2}$,

(b) $\beta(2(1-t)-1) = \beta(1-2t)$ if $(1-t) \ge \frac{1}{2}$ or equivalently if $t \le \frac{1}{2}$.

Similarly, we see that $\{(-\beta) + (-\alpha)\}(t)$ is given by

(c) $-\beta(2t) = \beta(1-2t)$ if $t \le \frac{1}{2}$,

(d) $-\alpha(2t-1) = \alpha\big(1-(2t-1)\big) = \alpha(2-2t)$ if $t \ge \frac{1}{2}$.

Therefore we have shown that for each $t \in [0,1]$ the values of $-(\alpha+\beta)$ and $(-\beta) + (-\alpha)$ at t are the same, and hence the two curves are equal.■

4. Let $i : A \to X$ denote the inclusion map. Since the fundamental group construction defines a covariant functor, we have $r_* \circ i_* = \mathrm{id}_{\pi_1(A)}$. Therefore if $u \in \pi_1(A, a_0)$ we have $u = r_*(i_*(u))$ and therefore u lies in the image of r_*, which means that r_* is surjective.■

7. (a) In order to prove functional identities, one needs to show that the values of both sides of the equation at every point s in the domain are the same. We apply this to verify the associativity, neutral element and inverse identities in $\Omega(G, 1)$:

Associativity. For all s we have

$$\{(f \otimes g) \otimes h\}(s) = \big(f(s) \cdot g(s)\big) \cdot h(s) = f(s) \cdot \big(g(s) \cdot h(s)\big) = \{f \otimes (g \otimes h)\}(s) .$$

Neutral element. If $C_1(t) = 1$ for all t, then for all s we have

$$\{f \otimes C_1\}(s) = f(s) \cdot 1 = f(s) , \qquad \{C_1 \otimes f\}(s) = 1 \cdot f(s) = f(s) .$$

Inverses. If $g(t) = f(t)^{-1}$ for all t, then for all s we have

$$\{f \otimes g\}(s) \;=\; f(s)\cdot g(s) \;=\; 1 \;=\; C_1(s)\ , \qquad \{g \otimes f\}(s) \;=\; g(s)\cdot f(s) \;=\; 1 \;=\; C_1(s)\ .$$

(b) The crucial point to verify is that if f_0 and g_0 are endpoint preserving homotopic to f_1 and g_1 respectively, then $f_0 \otimes g_0$ is endpoint preserving homotopic to $f_1 \otimes g_1$. If we know this, then we can define a binary operation on $\pi_1(G,1)$ by noting that there is a well defined binary operation on the latter with $[f] \otimes [g] = [f \otimes g]$. The associativity, neutral element and inverse identities will then follow from the corresponding identities derived in (a).

To prove the statement in the preceding paragraph, note that if H and K are endpoint preserving homotopies from f_0 and g_0 to f_1 and g_1 respectively, then $H \otimes K$ is endpoint preserving homotopy from $f_0 \otimes g_0$ to $f_1 \otimes g_1$.■

(c) Follow the hint. Direct computation yields the identity

$$f \;+\; g \quad=\quad (f \;+\; C_1) \otimes (C_1 \;+\; g)$$

from which we find that $[f]\cdot[g] = [f + C_1] \otimes [C_1 + g] \;=\; [f] \otimes [g]$.■

(d) For each value of s either $\{f + C_1\}(s)$ or $\{C_1 + g\}(s)$ is equal to 1, so these two curves commute with respect to the "$\otimes$" operation. Once again applying the reasoning in (c), we find that $[f] \otimes [g] = [g] \otimes [f]$ for all $[f]$ and $[g]$. The main conclusion of (c) now implies that $[f[\cdot[g] = [f] \otimes [g] = [g] \otimes [f] = [g]\cdot[f]$.■

Problems from Hatcher, pp. $38 - 40$

10. This is very similar to parts of the preceding exercise. Let $i_X : (X, x_0) \to (X \times Y, (x_0, y_0))$ be the slice inclusion sending x to (x, y_0), and let $i_Y : (Y, y_0) \to (X \times Y, (x_0, y_0))$ be the slice inclusion sending y to (x_0, y). The goal is to construct a homotopy from $(i_X \circ f) + (i_Y \circ g)$ to $(i_Y \circ g) + (i_X \circ f)$. As is always the case with mappings into products, it is enough to construct the homotopies for the coordinate projections of these maps onto (X, x_0) and (Y, y_0). In other words, we only need to construct homotopies

$$p_X \circ \big((i_X \circ f) + (i_Y \circ g) \big) \ \text{ to } \ p_X \circ \big((i_Y \circ g) + (i_X \circ f) \big)$$

$$p_Y \circ \big((i_X \circ f) + (i_Y \circ g) \big) \ \text{ to } \ p_Y \circ \big((i_Y \circ g) + (i_X \circ f) \big)$$

(where p_X and p_Y are coordinate projections), and we shall explain how these may be found using homotopies we have already constructed (for our puposes, this is "explicit" enough).

Since $p_X \circ i_X = \mathrm{id}_X$, $p_Y \circ i_Y = \mathrm{id}_Y$, $p_X \circ i_Y = \mathrm{constant}(x_0)$ and $p_Y \circ i_X = \mathrm{constant}(y_0)$, we can translate the display to conclude that we only need to construct homotopies from $f + \mathrm{constant}(x_0)$ to $\mathrm{constant}(x_0) + f$ and from $g + \mathrm{constant}(y_0)$ to $\mathrm{constant}(y_0) + g$. This can be done by splicing together the standard homotopies from $h + \mathrm{constant}$ to h and from h to $\mathrm{constant} + h$ for $h = f$ or g.■

13. As stated, the problem has an almost trivial solution which require NO HYPOTHESES on the map of fundamental groups or the arcwise connectedness of A. Here is the solution: Given any curve $\gamma : [0,1] \to X$ with endpoints in A, the homotopy $h_s(t) = \gamma(st)$ defines a homotopy from the constant curve with value $\gamma(0) \in A$ to the original curve γ. Presumably the author intended the following, which we shall prove below: ... iff every path in X with endpoints in A is *endpoint preserving* homotopic to a curve in A.

($\Longrightarrow$) Suppose that $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is onto, and let $a_0, a_1 in A$. Since A is arcwise connected, there are continuous curves α_0 and α_1 joining x_0 to a_0 and a_1 respectively.

We claim that every curve $\gamma : [0,1] \to X$ such that $\gamma(i) = a_i$ for $i = 0,1$ is endpoint preserving homotopic to a curve of the form $(-\alpha_0 + \beta) + \alpha_1$, where β is a basepoint preserving closed curve in X. In fact, if we set β equal to $(\alpha_0 + \gamma) + (-\alpha_1)$, then we have the identities

$$[\gamma] \quad = \quad [C(a_0) + \gamma + C(a_1)] \quad = \quad [(-\alpha_0) + \alpha_0 + \gamma + (-\alpha_1) + \alpha_1] \quad = \quad [(-\alpha_0 + \beta) + \alpha_1]$$

when we adopt the convention in the course notes that endpoint preserving homotopy classes of iterated concatenations do not depend upon where parentheses are inserted. The hypothesis in this part of the problem is that β is endpoint preserving homotopic to a curve θ whose image lies in A, and therefore $[\gamma] = [(-\alpha_0) + \beta + \alpha_1]$ is equal to

$$(-\alpha_0 + \theta) + \alpha_1$$

, where the image of the representative curve $(-\alpha_0 + \theta) + \alpha_1$ is contained in A.$\blacksquare$

($\Longrightarrow$) This follows immediately, for if γ is a closed curve in X which is endpoint (hence basepoint) preserving homotopic to a curve θ whose image is in A, then $[\gamma] = i_*([\theta])$.$\blacksquare$

Additional exercises

1. Parts of this exercise are similar to parts of Exercise 52.7 in Munkres, for which a solution is given above.

(i) The first statement to prove is that the map sending (α, β) to $\alpha + \beta$ is continuous. We can do this by showing that $\mathbf{d}(\alpha' + \beta', \alpha + \beta) < \varepsilon$ if $\mathbf{d}(\alpha', \alpha) < \varepsilon$ and $\mathbf{d}(\beta', \beta) < \varepsilon$. In fact, the maximum distance between the points $\{\alpha' + \beta'\}(s)$ and $\{\alpha + \beta\}(s)$ is the greater of (a) the maximum distance between the points $\{\alpha'\}(s)$ and $\{\alpha\}(s)$, (b) the maximum distance between the points $\{\beta'\}(s)$ and $\{\beta\}(s)$.

We now turn to the homotopy assertions, and we shall use the notation of Proposition VIII.1.3. Let $H_R : [0,1] \times [0,1] \to [0,1]$ be the straight line homotopy from the identity to h_R, and let $H_L : [0,1] \times [0,1] \to [0,1]$ be the straight line homotopy from the identity to h_L. Define corresponding maps $K_R, K_L : \Omega(X, x) \times [0,1] \to \Omega(X, x)$ by $\{K_R(\gamma, t)\}(s) = \gamma \circ H_R(s, t)$ and $\{K_L(\gamma, t)\}(s) = \gamma \circ H_L(s, t)$. The definitions then imply that $K_L(\gamma, 0) = \gamma \circ h_L = C_x + \gamma$ and $K_L(\gamma, 1) = \gamma$, and similarly we know that $K_R(\gamma, 0) = \gamma \circ h_R = \gamma + C_x$ and $K_R(\gamma, 1) = \gamma$. If γ' is another element of $\Omega(X, x_0)$, then by construction the distance between $K_R(\gamma, t)$ and $K_R(\gamma', t)$ is equal to the distance between γ and γ', and likewise for $K_R(\gamma, t)$ and $K_R(\gamma', t)$. These imply that K_L and K_R are continuous and hence yield homotopies of the types described in the statement of the exercise.

(ii) Follow the hint and imitate the reasoning in Munkres, Exercise 52.7. The results of (i) show that $(Y, e) = (\Omega(X, x), C_x)$ is an H-space as defined following the statement of this exercise. We shall prove more generally that $\pi_1((Y, e)$ is abelian using the approach for the exercise in Munkres. Let $m : Y \times Y \to Y$ denote the continuous binary operation, and define a binary operation "$\otimes$" on $\pi_1(Y, e)$ sending $([f], [g])$ to the class of the composite $f \otimes g(s) \;=\; m\big(f(s), g(s)\big)$. We shall often denote the right hand side by $f(s) \cdot g(s)$ for the sake of simplicity, but when we do so we have to remember that this construction does not necessarily satisfy associativity or neutral element identities; all we know is that f is basepoint preservingly homotopic to both $f \cdot C_e$ and $C_e \cdot f$. The latter suffice to yield two weaker identities

$$[f \;+\; g] \quad = \quad [f \;+\; C_1] \otimes [C_1 \;+\; g] \;, \qquad [f \;+\; g] \quad = \quad [C_1 \;+\; g] \otimes [f \;+\; C_1]$$

and since the right hand sides are equal to $[f] \otimes [g]$ and $[g] \otimes [f]$ respectively, it follows that the "$\otimes$" operation agrees with the usual group operation on the fundamental group, and both operations are abelian.■

(*iii*) If $\varepsilon > 0$, then there is some $\delta > 0$ such that $\mathbf{d}(x, x') < \delta$ implies $\mathbf{d}(\, f(x),\ f(x')\,) < \varepsilon$. It follows that if α and β are curves such that $\mathbf{d}(\alpha, \beta) < \delta$, then $\mathbf{d}(f \circ \alpha, f \circ \beta) < \varepsilon$ (because the distance between two curves is the maximum distance between their values at points of the domain).

(*iv*) The second part follows from the first because continuous mappings from one compact metric space to another are always uniformly continuous, so everything reduces to proving the first assertion in the conclusion. For $n \geq 2$, define $\pi_n(X) = \pi_1(\Omega^{n-1}(X, x), \text{basepoint})$ as in the statement of the exercise. By (*ii*) we know this is an abelian group. By (*iii*) and induction we know that a basepoint preserving uniformly continuous mapping f induces a map with the same properties, say $\Omega^{n-1}(f)$, from $\Omega^{n-1}(X, x)$ to $\Omega^{n-1}(Y, y)$, and this construction is functorial because it is given by composition of functions. Define $f_* : \pi_n(X, x) \to \pi_n(Y, y)$ to be the homomorphism of fundamental groups induced by $\Omega^{n-1}(f)$.■

Note. Hatcher and most other books covering homotopy theory define π_n differently, but eventually one almost always finds a proof that their construction(s) is/are equivalent to the one given here.

2. We claim that basepoint preserving maps from $(S^0, 1)$ to (X, x) are the same as maps from $\{-1\}$ to X and basepoint preserving homotopies are the same as homotopies of such mappings $\{-1\} \to X$. The first part is true because the basepoint of S^0 must go to the basepoint of X, but there are no constraints on where the second point can go. To see the statement on homotopies, note that the restriction of a homotopy to $\{1\} \times [0, 1]$ must be constant but the restriction to $\{-1\} \times [0, 1]$ can be an arbitrary continuous mapping from $[0, 1]$ into X.■

VIII.2 : An important special case

Problems from Munkres, § 58, pp. 366 – 367

2. (*a*) Infinite cyclic.

(*c*) Infinite cyclic.

(*d*) Infinite cyclic.

(*f*) Infinite cyclic.

(*g*) Infinite cyclic.

(*h*) Trivial.

(*i*) Infinite cyclic.

(*j*) Infinite cyclic.

9. (*a*) *This was not included because our definition does not involve any choices of an initial point on the circle.*

(*b*) As in the statement of the exercise, let $\omega(t) = \exp 2\pi\, i\, t$, and let $t_0 \in \mathbb{R}$ be such that $p(t_0) = h \circ \omega(0)$, where $p : \mathbb{R} \to S^1$ is the usual map $p(t) = \exp 2\pi\, i\, t$. Let α be the unique path lifting of $h \circ \omega$ starting at t_0. Then *deg* (h) is the unique integer $d(h)$ such that $\alpha(1) = t_0 + d(h)$.

Now let $H : S^1 \times [0,1] \to S^1$ be a homotopy from h to k, and let $L : [0,1] \times [0,1] \to \mathbb{R}$ be the unique lifting of H such that $L(0,0) = t_0$ and $p \circ L(s,t) = H(p(s),t)$. By the uniqueness of path lifings we know that $\alpha(s) = L(s,0)$ and the curve $L(s,1)$ is a lifting β of k. Furthermore, since $L(1,t)$ and $L(0,t)$ are both liftings of the curve $H|\{1\} \times [0,1]$. it follows that there is some integer Δ such that $L(1,t) = L(0,t) + \Delta$ for all t. In particular, we have $L(1,1) = L(0,1) + \Delta$ and $L(1,0) = L(0,0) + \Delta$. Now $L(s,0) = \alpha(s)$ by the uniqueness of path liftings, and therefore $\Delta = deg\,(h)$. On the other hand, $L(s,1)$ is a lifting of $k(s) = H(s,1)$, and therefore it also follows that $\Delta = deg\,(k)$. In words, we have shown that (freely) homotopic maps from S^1 to itself have the same degree.■

(c) This one is probably easier to prove using the functoriality properties of the fundamental group. The proof of the main theorem in this section immediately yields the following result:

CLAIM. *If f is a basepoint preserving continuous mappings from S^1 to itself (with say $f(1) = 1$), then the self-homomorphism f_* on $\pi_1(S^1, 1) \cong \mathbb{Z}$ is multiplication by $deg\,(f)$.*

Proof of the claim. Since the class $[\omega]$ generates $\pi_1(S^1, 1)$, it is enough to prove that $f_*([\omega]) = deg\,(f) \cdot [\omega]$, and we can do this using the Path Lifting Property because the image $\partial(f)$ of $f_*([\omega]) = [f \circ \omega]$ in $\mathbb{Z}$ is given by taking a lifting θ of $f \circ \omega$ starting at $0 \in \mathbb{R}$ and setting $\partial(f)$ equal to $\theta(1)$. This coincides with the definition of degree for the mapping f.■

Before proceeding, we note that the claim yields the conclusion of the exercise if h and k are basepoint preserving, for in that case we have

$$\deg(h \circ k) \cdot [\omega]) \quad = \quad (h \circ k)_*([\omega]) \quad = \quad (h_* \circ k_*)([\omega]) \quad = \quad h_*\big(\deg(k) \cdot [\omega]\big) \quad =$$

$$\deg(k) \cdot h_*([\omega]) \quad = \quad \deg(k) \cdot \deg(h) \cdot [\omega]$$

and since $[\omega] \in \pi_1(S_1, 1) \cong \mathbb{Z}$ has infinite order it follows that $\deg(h \circ k) = \deg(h) \cdot \deg(k)$.

In order to apply the preceding discussion to h and k, we have to replace them with (freely) homotopic mappings which ARE basepoint preserving. The easiest way to do this is to choose a and b such that $p(a) = h(1)$ and $p(b) = k(1)$ and define homotopies by $H(z,t) = h(z) \cdot p(ta)^{-1}$ and $K(z,t) = k(z) \cdot p(tb)^{-1}$; it then follows that $H_0 = h$ and H_1 is basepoint preserving, and similarly $K_0 = k$ and K_1 is basepoint preserving. Applying (ii) and recalling that $H_1 \circ K_1$ is homotopic to $h \circ k$, we find that

$$\deg(h \circ k) \quad = \quad \deg(H_1 \circ K_1) \quad = \quad \deg(H_1) \cdot \deg(H_1) \quad = \quad \deg(h) \cdot \deg(k)$$

which is what we wanted to prove.■

Note. The tools developed in 205B yield a simpler proof which generalizes to a notion of degree for continuous self maps of S^n when $n \geq 2$.

(d) The unique lifting α of a constant map f is a constant map, so $\alpha(1) = \alpha(0)$ implies that the degree is zero. — For the identity, we know that ω is a unique lift, and here the degree is $\omega(1) - \omega(0) = 1$. — For z^n where n is an integer, we know that a unique lifting is given by $\omega_n(t) = nt$, and in this case $\omega_n(1) - \omega_n(0) = n$ is the degree; this applies to each of the final two cases in the exercise.■

(e) Suppose that h and k have the same degree, and let H_1 and K_1 be basepoint preserving maps which are freely homotopic to h and k respectively. Since $h \simeq H_1$ and $k \simeq K_1$, it suffices to show that if h and k have the same degree (so by (i) the same is true for H_1 and K_1), then H_1 and K_1 are homotopic. But the latter follows directly from the proof of the main result, for if the degrees are equal then $\partial(H_1) = \partial(K_1)$, and hence the unique liftings of H_1 and K_1 starting at 0 are endpoint preserving homotopic. We can then compose this homotopy with p to obtain a homotopy from H_1 to K_1, and since homotopy of maps is an equivalence relation it will follow that $h \simeq k$.■

Problems from Hatcher, pp. $38 - 40$

3. ($\Longrightarrow$) Suppose that X is arcwise connected and $\pi_1(X, x_0)$ is abelian. If α and β are two continuous curves joining x_0 to $x_1 \in X$, we want to prove that $\alpha^* = \beta^*$. — The desired conclusion is equivalent to proving that $(\alpha + (-\beta))^*$ induces the identity on $\pi_1(X, x_0)$ for all α and β, and if we make the substitution $\gamma = \alpha + (-\beta)$ this translates into proving that for each closed curve γ which starts and ends at x_0 the automorphism γ^* — which we have shown is given by $\gamma^*(u) = [\gamma]^{-1}\, u\, [\gamma]$ — is the identity. This identity holds if and only if $\pi_1(X, x_0)$ is abelian, so $\alpha^* = \beta^*$ in this case.

($\Longrightarrow$) Suppose now that we have $\alpha^* = \beta^*$ for all paths α and β from x_0 to x_1. By the same reasoning as in the first part, it follows that $\gamma^*(u) = [\gamma]^{-1}\, u\, [\gamma]$ is the identity for all closed curves γ and $u \in \pi_1(X, x_0)$. In other words, we have $u = v^{-1}uv$ for all u and v in the fundamental group. But this is true if and only if $\pi_1(X, x_0)$ is abelian.■

16. If $A \subset X$ and a retraction $r : X \to A$ exists, then the induced map in fundamental groups $\pi_1(A) \to \pi_1(X)$ is 1–1 and the induced map r_* in fundamental groups is onto, so it suffices to prove that either $\pi_1(A) \to \pi_1(X)$ is not 1–1 or $\pi_1(A)$ is not isomorphic to a homomorphic image of $\pi_1(X)$.

(a) If $X = \mathbb{R}^3$ and $A \cong S^1$ then $\pi_1(X)$ is trivial but $\pi_1(A) \cong \mathbb{Z}$. Since a nontrivial group cannot be isomorphic to a homeomorphic image of a trivial group, there cannot be a retraction.■

(b) This is a little informal, in part because the inclusion $A \to X$ is only described by means of an unannotated drawing, but it can be justified if one describes the inclusion $A \subset X$ explicitly; this turns out to be possible, but at this stage it would take several paragraphs to explain everything in full detail. In this case both $\pi_1(A)$ and $\pi_1(X)$ are infinite cyclic, and if $B \subset X = S^1 \times D^2$ is equal to $S^1 \times \{\mathbf{0}\}$, then the inclusion of $A \cong S^1$ in X is homotopic to the map $S^1 \to B \cong S^1$ given by something like $p \circ f(t)$, where $f : [0,1] \to \mathbb{R}$ is the closed curve given by $f(t) = (1+h)\sin t$ for some small $h > 0$. By convexity f is homotopic to a constant curve in $\mathbb{R}$, so $p \circ f$ is homotopic to a constant curve in B, and hence the mapping $A \to B$ is null homotopic. But this means that the composite $\pi_1(A) \to \pi_1(B) \cong \pi_1(X)$ is trivial. Since $\pi_1(A)$ and $\pi_1(X)$ are infinite cyclic, this means that the map $\pi_1(A) \to \pi_1(X)$ is not 1–1, and we have noted that in such cases a retraction cannot exist.■

17. The assignment modifies the question to ask for infinitely many homotopy classes of retractions for the slice inclusion $j_1 : S^1 \to S^1 \times S^1$ sending z to $(z, 1)$. At the end of the exercise we shall explain how one can extract a solution to the exercise as it is stated in Hatcher.

Given an integer n, let $r_n : S^1 \times S^1 \to S^1$ be the map sending (z, w) to $z\,w^n$. It follows that for each n we have $r_n \circ j_1 = \mathrm{id}$. To prove that r_n and r_m are not homotopic if $m \neq n$, let $j_2 : S^1 \to S^1 \times S^1$ be the other slice inclusion $j_2(z) = (1, z)$. Then the degree of $r_n \circ j_2$ is equal to n, and therefore the homotopy classes of the mappings r_n and r_m are distinct if $m \neq n$.■

Solution for the problem as stated in Hatcher. In this case $S^1 \vee S^1$ is identified with the subspace

$$S^1 \times \{1\} \;\cup\; \{1\} \times S^1 \quad \subset \quad S^1 \times S^1$$

and the slice inclusions j_1, j_2 factor through maps i_1, i_2 from S^1 to $S^1 \vee S^1$. Therefore if $r_n' = r_n | S^1 \vee S^1$ we have $r_n' \circ i_1 = \mathrm{id}$ and $\deg\,(r_n' \circ i_2) = n$.■

Additional exercises

0. This was done already as a step in the solution to Munkres, Exercise 58.9 (*q.v.*).■

1. Follow the hint. Given continuous mappings $f, g : S^1 \to S^1$, with product $h(z)$, let $\alpha, \beta : [0,1] \to \mathbb{R}$ be liftings and let γ be their algebraic sum (if we write this symbolically, it will conflict with the use of "+" for concatenation); as suggested by the hint, the curve γ is a lifting of h. By definition the degree of h is equal to $\gamma(1) - \gamma(0)$, and the latter is equal to

$$\alpha(1) + \beta(1) - \alpha(0) - \beta(0) \quad = \quad \big(\,\alpha(1) - \alpha(0)\,\big) + \big(\,\beta(1) - \beta(0)\,\big) \quad = \quad \deg(f) + \deg(g)$$

which is what we wanted to prove.■

2. (*i*) It is probably best to start by describing $\Pi(X)$ more explicitly. Its objects are the points of X, and a morphism from x_0 to x_1 is an endpoint preserving homotopy class of curves from x_0 to x_1. The identity morphism for x is just the homotopy class of the constant curve at x, and if α and β are curves joining x_0 to x_1 and x_1 to x_2 respectively, then the formal composite $[\beta] \circ [\alpha]$ is merely $[\alpha + \beta]$ (note the order reversal!). The morphisms from x to itself then correspond to elements of $\pi_1(X, x)$ (with reversed multiplication!), and $[\alpha]^{-1} = [-\alpha]$ by the results of Sections VIII1 and VIII.2.

Functoriality can now be seen fairly easily. On objects, each point $x \in X$ goes to $f(x) \in Y$, and the homotopy class of a curve α joining x_0 to x_1 goes to the homotopy class $f_*([\alpha])$ of $f \circ \alpha$. The results of this and the previous section imply that this does not depend on which α we choose to represent an endpoint preserving homotopy class. Since $f \circ \alpha$ is constant if α is constant, it follows that the identity morphism for x is sent to the identity morphism for $f(x)$, and the chain of identities

$$[f \circ \beta]\,[f \circ \alpha] \quad = \quad [f \circ (\alpha + \beta)] \quad = \quad f_*([\alpha + \beta]) \quad = \quad f_*([\beta][\alpha])$$

shows that the construction preserves composition of morphisms.■

3. Before starting, we recall the construction of $J_k : \pi_1(T^k, \mathbf{1}) \to \mathbb{Z}^k$: Let $p_k : \mathbb{R}^k \to T^k$ be the Cartesian k^{th} power map $p \times \; \cdots \; \times p$ (k factors) from $\mathbb{R}^k$ to T^k. Let $\mathbf{1} \in T^k$ be the point whose coordinates are all equal to the unit element of S^1.

If γ is a basepoint preserving closed curve in T^k which starts and ends at 1, the we can apply the Path Lifting and Covering Homotopy Properties to each coordinatefunction of γ; this yields a unique lifting of γ to a curve $\Gamma : [0,1] \to \mathbb{R}^k$ such that $\Gamma(0) = \mathbf{0}$, and $\Gamma(1) \in \mathbb{Z}^n$ because $p \circ \Gamma(1) = \mathbf{1}$. As in the proof of the main result, $\Gamma(1)$ only depends upon the basepoint preserving homotopy class of γ, and hence the map $\gamma \to \Gamma(1)$ defines a map $J_k : \pi_1(T^k, \mathbf{1}) \to \mathbb{Z}^k$; the argument proving the main result of this section also shows that J_k is a group isomorphism. The inverse map is given by sending $(d_1, \; \cdots \; , d_k)$ to the homotopy class of the curve $(p(d_1 t), \; \cdots \; , p(d_k t))$, where $p : \mathbb{R} \to S^1$ is the usual map.

(*i*) If $(a, b) \in \mathbb{Z}^2$, then $m_* \circ J_2^{-1}$ sends (a, b) to the curve $p(at) \cdot p(bt) = p((a+b)t)$, and J_1^{-1} takes the latter curve to $a + b \in \mathbb{Z}^2$.■

(*ii*) Suppose that f is given as in the exercise. For each j between 1 and k let $s_j : S^1 \to T^k$ denote the slice inclusion onto the j^{th} slice (all coordinates are 1 except possibly the j^{th} one). Then the images of the homomorphisms s_{j*} generate $pi_1(T^k, \mathbf{1})$, so it is enough to compute the degree of each map $f \circ s_j$. The latter is the map sending z to z^{c_j}, and therefore the degree is equal to c_j. This immediately yields the conclusion of part (*ii*).■

4. (*i*) Since we are mapping into a product, it is enough to prove this for projection onto the factors, and we have already done this in the second part of the preceding exercise.■

(ii) If Φ_A is a homeomorphism, or even homotopic to a homeomorphism, then the induced homomorphism Φ_{A*} of $\pi_1(T^k, \mathbf{1}) \cong \mathbb{Z}^k$, which corresponds to left multiplication by A on $\mathbb{Z}^k$, must be an automorphism, and therefore $\det A$ must be equal to ± 1. Conversely, if $\det A = \pm 1$ and $B = A^{-1}$, then by Cramer's Rule B has integral entries and hence we can construct Φ_B. The definitions of the maps imply that $\Phi_{AB} = \Phi_A \circ \Phi_B$ and similarly if B and A are reversed; furthermore, if A is the identity matrix I then it follows that Φ_I is the identity mapping. These observations combine to imply that Φ_B is inverse to Φ_A.■

(iii) The first part follows because an arbitrary self-homomorphism (*endomorphism*) of the group $\pi_1(T^k, \mathbf{1}) \cong \mathbb{Z}^k$ is given by some $k \times k$ matrix A with integral entries, and we have shown that Φ_{A*} corresponds to left multiplication by A. The second part follows because if this matrix A is 1–1 and onto then Φ_A is a basepoint preserving homeomorphism from T^k to itself.■

VIII.3 : Covering spaces

Problems from Munkres, § 53, *p.* 341

For these exercises, we do NOT assume the Default Hypothesis.

1. If p is given as in the exercise, then for each $x \in X$ the whole space X is an evenly covered open neighborhood of x in X.■

2. We know that $p * -1[U]$ is homeomorphic to $U \times F$ for some discrete space F. Since U is connected, this means that the sheets $U \times \{y\}$ are the connected components of $U \times F$, and under the homeomorphism they correspond to the connected components of $p * -1[U]$.■

4. Let $z \in Z$, and let U be an open neighborhood of $z \in Z$ which is evenly covered; specifically, let $r^{-1}[U]$ be the union of the pairwise disjoint open subsets $U_1, \cdots, U_n$ such that r maps each U_j homeomorphically onto U, and let $y_j \in U_j$ be the unique point such that $r(y_j) = z$. Since q is a covering space projection, for each j there is an open neighborhood V_j of y_j in Y which is evenly covered. Intersecting V_j with U_j if necessary, we might as well assume that $V_j \subset U$ (an open subset of an openly covered open set is also evenly covered!). Let $W = \cap_j \, r[V_j]$. Then by construction $z = r(y_j)$ (for each j) lies in W, and this set is open in Z because r is an open mapping. Furthermore, W is evenly covered by the union of the pairwise disjoint subsets W_j. Now let $W_j = r^{-1}[W] \cap V_j$, so that $y_j \in W_j$ and W_j is also evenly covered. Therefore, for all j the inverse image $q^{-1}[W_j]$ is homeomorphic to $W_j \times F_j$, where F_j is discrete, such that the restriction of q to $q^{-1}[W_j]$ corresponds to projection onto W_j.

For each j let $r_j : W_j \to W$ denote the homeomorphism determined by r. Since $p = r \circ q$, it follows that

$$p^{-1}[W] \quad = \quad q^{-1}\left[r^{-1}[W]\right] \quad = \quad q^{-1}\left[\bigcup_j W_j\right] \quad \cong$$

$$\coprod_j W_j \times F_j \quad \cong \quad W \times \left(\coprod_j F_j\right)$$

where we use the homeomorphisms $W_j \cong W$ at the last step. This implies that the open neighborhood W of z is evenly covered with respect to p, and the latter means that p is also a covering space projection.■

6. (a) Let $p : X \to Y$ be a covering space projection. This part of the exercise involves proving that if Y has a stated topological property, then so does X. There are several distinct properties, and we shall verify the assertions about them separately. Note that the priorities on the parts of this exercise vary depending upon the specific property.

Hausdorff. Let $x_1 \neq x_2$ in X. There are two cases depending upon whether or not $p(x_1) = p(x_2)$. If $p(x_1) \neq p(x_2)$, then since Y is Hausdorff there are disjoint open neighborhoods U_1 and U_2 of $p(x_1)$ and $p(x_2)$ in Y, and their inverse images $p^{-1}[U_1]$ and $p^{-1}[U_2]$ are disjoint open neighborhoods of x_1 and x_2 in X. On the other hand, if $p(x_1) = p(x_2)$, let W be an evenly covered open neighborhood of this point. Then $p^{-1}[W]$ is an open set homeomorphic to a disjoint union of copies of W. Since $x_1 \neq x_2$, one of these copies contains x_1 and another contains x_2, and these two copies of W in X are disjoint open neighborhoods of the two points.■

Regular. Let $x \in X$. It suffices to prove the regularity condition for a set of open neighborhoods $\mathcal{V}$ of x such that every open neighborhood of X contains a subneighborhood in $\mathcal{V}$, for if $x \in W$ open and $x \in V \subset W$ with $V \in \mathcal{V}$, then $x \in U \subset \overline{U} \subset V$ implies the same inclusions with W replacing V.

In view of the preceding discussion, let $x \in V$, where V is an open neighborhood of x such that $p[V] = W$ is evenly covered and V is one of the sheets. Since Y is regular, there is an open neighborhood U of $p(x)$ such that $p(x) \in U \subset \overline{U} \subset W$. Let $U_1 = p^{-1}[U] \cap V$ be the sheet over U which is contained in V. If we can prove that the closure $\overline{U_1}$ of U_1 in X is contained in V, then the defining condition for regularity will hold at the point x, and since x was chosen arbitrarily this will imply that X is regular.

It will suffice to prove that $p^{-1}\left[\overline{U}\right] \cap V$, which contains U_1, is closed in X because we would then have $\overline{U_1} \subset p^{-1}\left[\overline{U}\right] \cap V \subset V$. By construction and continuity we know that $p^{-1}\left[\overline{U}\right]$ is closed in X and it is contained in the evenly covered open subset $p^{-1}[W]$. If V^* denotes the union of all sheets in $p^{-1}[W]$ except X, then V^* is open and hence $X - V^*$ is closed in X; with this notation we can rewrite the inclusion in the preceding sentence in the form $p^{-1}\left[\overline{U}\right] \subset V \cup V^*$, and it follows that

$$p^{-1}\left[\overline{U}\right] \cap V \quad = \quad p^{-1}\left[\overline{U}\right] \cap (X - V^*)$$

and since the right hand side is an intersection of two closed subsets of X, it follows that the left hand side is also a closed subset of X, which is what we needed to complete the proof.

Completely regular. By the preceding discussion we know that X is regular, and as in the preceding discussion it suffices to prove that if V is an open neighborhood of x such that $p[V] = W$ is evenly covered and V is one of the sheets. Let U be an open subneighborhood such that $x \in U \subset \overline{U} \subset V$, so that $U' = p[U]$ and $V' = p[V]$ are open neighborhoods of $p(x)$ such that $p(x) \in U' \subset \overline{U'} \subset V'$. Since a subspace of a completely regular space is regular, there is a continuous function $g : V' \to [0,1]$ such that $g(p(x)) = 0$ and $g = 1$ on $V' - U'$; the composite $g \circ p$ satisfies similar properties: The value of $g \circ p$ at x is zero and $g \circ p = 1$ on $V - U$. If we define a new function $f : Xto[0,1]$ such that $f|\overline{U} = g \circ p|\overline{U}$ and $f|X - U = 0$, then these functions agree on the overlapping closed subset $\overline{U} - U$, and therefore f defines a continuous function on X which is 1 at x and 0 off U.■

Locally compact Hausdorff. We already know that X is Hausdorff, so it is only necessary to show that every $x \in X$ has some open neighborhood whose closure is compact. As before, let V be an open neighborhood of x such that $p[V] = W$ is evenly covered and V is one of the sheets. Since Y is locally compact, it follows that there is some open neighborhood U of $p(x)$ such that $\overline{U} \subset V$ and $\overline{U}$ is compact. Let $U_1 = p^{-1}[U] \cap V$ be the sheet over U which is contained in V, and let $F = p^{-1}\left[\overline{U}\right] \cap V$. Since $p|V$ is a homeomorphism onto an open subset, it follows that F is homeomorphic to $\overline{U}$ and hence is compact. Therefore we have found an open neighborhood U_1 of x and a compact subset $F \subset X$ such that $x \in U_1 \subset F \subset V$. Since X is compact we know that F is closed in X, and therefore it follows that $\overline{U_1}$ is also compact. As before, we started with an arbitrary $x \in X$, so the argument implies that X is locally compact near every point and hence is a locally compact (Hausdorff) space.■

(b) We shall first prove this when Y is Hausdorff (in which case X is also Hausdorff by part (a).

Let $p : X \to Y$ be a covering space projection such that Y is compact and there are only finitely many sheets at each point of Y. We shall prove that X is a union of finitely many compact subsets. Let $y \in Y$, and let V_y be an open neighborhood of y which is evenly covered. Since a locally compact Hausdorff space is regular, there is some subneighborhood W_y of V_y such that $\overline{W_y} \subset V_y$ (and $\overline{W_y}$ is compact). The family $\mathcal{W}$ of all sets W_y is an open covering of Y, so there is a finite subcovering $\{W_{y_1}, \cdots, W_{y_k}\}$. Since the covering has finitely many sheets, each of the sets $p^{-1}\left[\overline{W_{y_j}}\right]$ is also compact, and since these form a finite closed covering of X it follows that X is also compact.

Here is a proof when Y is not necessarily Hausdorff.

Let $\mathcal{U}$ be an open covering of X. For each $y \in Y$, let V_y be an evenly covered open subset, and let $V_{y,1}, \cdots V_{y,k(y)}$ denote the sheets over V_y (we know there are only finitely many). If $x_j \in V_{y,j}$ is the unique point such that $p(x_j) = y$, then there is an open subneighborhood $\Omega_{y,j}$ of x_j which is contained in some open set U_α in the open covering $\mathcal{U}$. Since p is an open mapping, the set $W_y = \cap_j \, p\left[\Omega_{y,j}\right]$ is an evenly covered open neighborhood of y which is contained in V_y. If $W_{y,j} = p^{-1}[W_y] \cap V_{y,j}$, then p maps this set homeomorphically onto the open neighborhood W_y.

Since Y is compact, the open covering $\mathcal{W}$ of Y by the open subsets W_y has a finite subcovering consisting of sets W_z, where z lies in some finite subset $Z \subset Y$. The sheets of the inverse images $p^{-1}[W_z]$ then form an open covering of X such that each set in this open covering is contained in some U_α which belongs to $\mathcal{U}$. Therefore if for each $W_{z,j}$ we choose some $U_{\alpha(z,j)}$ such that $W_{z,j} \subset U_{\alpha(z,j)}$, then the open sets $U_{\alpha(z,j)}$ form a finite subcovering of $\mathcal{U}$.■

Problem from Hatcher, pp. $79 - 82$

2. Assume that we are given covering space projections $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$; we need to prove that $p_1 \times p_2$ is also a covering space projection.

Let $(x_1, x_2) \in B_1 \times B_2$. Then the hypotheses imply that for $i = 1, 2$ there is an evenly covered open neighborhood U_i of X_i; *i.e.*, there are discrete spaces A_i and B_i together with homeomorphisms $h_i : p_i^{-1}[U_i] \to U_i \times A_i$ such that $\mathrm{proj}(U_i) \circ h_i$ is the restriction of p_i to $p_i^{-1}[U_i]$. We claim that $U_1 \times U_2$ is an evenly covered neighborhood of (x_1, x_2) in $B_1 \times B_2$. This is true because the homeomorphism

$$H : (p_1 \times p_2)^{-1}[U_1 \times U_2] \;=\; p_1^{-1}[U_1] \times p_2^{-1}[U_2] \;\longrightarrow\; U_1 \times A_1 \times U_2 \times A_2 \;\cong\; (U_1 \times U_2) \times (A_1 \times A_2)$$

(where the last map switches the second and third factors) is such that $\mathrm{proj}(U_1 \times U_2) \circ H$ is the restriction of $p_1 \times p_2$ to $(p_1 \times p_2)^{-1}[U_1 \times U_2]$.■

1. (i) Let U be an open subset of B, and consider the following commutative diagram, which is derived from the exercise by considering various restriction mappings:

$$\begin{array}{ccc} p^{-1}[U] & \xrightarrow{\Phi_U} & f^{-1}\left[\varphi[U]\right] \\ \downarrow p & & \downarrow f_{\varphi[U]} \\ U & \xrightarrow{\varphi_U} & \varphi[U] \end{array}$$

Let $y \in Y$, set x equal to $\varphi^{-1}(y)$, and let U be an evenly covered open neighborhood of x, so that there is a homeomorphism $h : p^{-1}[U] \to U \times A$ satisfying the defining identity. Then the composite

$$h' \quad = \quad (\varphi_U \times \mathrm{id}) \circ h \circ \Phi_U^1$$

is a homeomorphism from $f^{-1}\left[\varphi[U]\right]$ to $\varphi[U] \times A$ such that $f_{\varphi[U]}$ is h' followed by coordinate projection onto $\varphi[U]$. Therefore $\varphi[U]$ is an evenly covered open neighborhood of y.■

(ii) Let $x \in X$, and choose U_α in $\mathcal{U}$ such that $x \in U_\alpha$. Since q_α is a covering space projection, we know that there is some open neighborhood $V \subset U_\alpha$ of x such that V is evenly covered by q_α, and since q_α has the same values as p at all points (but a different domain and codomain) it follows that V is also evenly covered by p.■

2. Given $x \in X$ we can find open neighborhoods U_1 and U_2 of x which are evenly covered by p_1 and p_2, and by local connectedness there is a connected subneighborhood $U \subset U_1 \cap U_2$; it follows that U is evenly covered with respect to both p_1 and p_2. If $\mathcal{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ is an open covering of E_2 by open subsets which are connected and evenly covered by p_1 and p_2, then by part (ii) of Exercise 1 it will suffice to show that the restricted maps

$$p_1^{-1}[U_\gamma] = p^{-1}\left[p_2^{-1}[U_\gamma]\right] \;\longrightarrow\; p_2^{-1}[U_\gamma]$$

determined by p are all covering space projections.

If U is connected and evenly covered by p_1 and p_2, let Let $h_1 : p_1^{-1}[U] \to U \times A$ and $h_2 : p_2^{-1}[U] \to U \times B$ be homeomorphisms (where A and B are discrete spaces) such that $\mathrm{proj}(U) \circ h_i$ is the restriction of p_i, and let $q : U \times A \to U \times B$ be the map unique continuous mapping from $p_1^{-1}[U] = p^{-1}\left[p_2^{-1}[U]\right]$ to $p_2^{-1}[U]$ such that $p\left(h_1^{-1}(u,a)\right) = h_2^{-1}\left(q(u,a)\right)$. By continuity, for each $a \in A$ the map q sends the connected subset $U \times \{a\} \subset U \times A$ into some connected component $U \times \{b(a)\} \subset U \times B$, and q is onto because p is onto. Since the composites

$$U \times \{a\} \;\longrightarrow\; p_1^{-1}[U] \;\longrightarrow\; U, \quad U \times \{b\} \;\longrightarrow\; p_2^{-1}[U] \;\longrightarrow\; U$$

are homeomorphisms, it follows that q must map $U \times \{a\}$ homeomorphically onto $U \times \{b(a)\}$; denote the corresponding homeomorphism from U to itself by q_a. If we compose h with the union of the homeomorphisms $\varphi = \cup_a \, q_a : U \times A \to U \times A$ and replace h with the composite $\varphi \circ h$, then we obtain a new map q' analogous to q but satisfying the condition that q maps $U \times \{a\}$ to $U \times \{b(a)\}$ to the identity; note that q' is onto because q is onto. It follows that q' is a covering space projection, and by part (i) of Exercise 1 it also follows that the map

$$p_1^{-1}[U] = p^{-1}\left[p_2^{-1}[U]\right] \;\longrightarrow\; p_2^{-1}[U]$$

determined by p is also a covering space projection.

If we apply the reasoning of the preceding paragraph to the open covering of E_2 by the sets $p_2^{-1}[U_\gamma]$, where U_γ belongs to the previously described open covering $\mathcal{U}$, we see that each of the maps

$$p_1^{-1}[U_\gamma] = p^{-1}\left[p_2^{-1}[U_\gamma]\right] \longrightarrow p_2^{-1}[U_\gamma]$$

is a covering space projection, and as noted above this suffices to prove that p itself is a covering space projection.

Note. Here is an example where p is not onto: Let X be an arbitrary nonempty space. Take $p : X \to X \amalg X$ to be inclusion into the first summand, and take $p_2 : X \amalg X \to X$ to be the map which is the identity on each summand. There is a variation of this exercise in which one replaces the surjectivity hypothesis on p by an assumption that E_2 is connected; the relationship with Exercise 2 is that if E_2 is connected then one can prove that p is onto.

3. (i) Let $y \in Y$, let U be an evenly covered open neighborhood of $f(x)$, and let V be an open neighborhood of x such that $f[V] \subset U$. We claim that V is evenly covered in $Y \times_X E$. The definitions imply that the inverse image of V is the subspace of $V \times E$ defined by $f(v) = p(e)$, and the inverse image is also the subspace of $V \times p^{-1}[U]$ defined by the same equation. If h is the homeomorphism $p^{-1}[U] \to U \times A$ which exists by the assumption that U is evenly covered, then under this homeomorphism the inverse image of V corresponds to the set of all points (v, u, a) in $V \times U \times A$ such that $f(v) = u$; *i.e*, the inverse image is given by the product of A with the graph of $f|U$. Since the graph is homeomorphic to U, it follows that the inverse image is homeomorphic to U via coordinate projection; one can check directly that all these maps are compatible with the appropriate projections onto U, and this shows that U is evenly covered.

For the second part, if a lifting φ exists, then the image of the map sending y to $(y, \varphi(y))$ is contained in $Y \times_X E$ and hence determines a map $s : Y \to Y \times_X E$; by definition the composite of s with projection onto Y is the identity. Conversely, if there is a map $s : Y \to Y \times_X E$ such that $p_{(Y,f)}s = 1_Y$, then the composite of s with $j : Y \times_X E \subset E$ satisfies $p \circ j \circ s = f$, and therefore $j \circ s$ is a lifting of f.∎

(ii) If f is a subspace inclusion then $Y \times_X E$ is just the set of all points (y, e) in $Y \times p^{-1}[Y]$ such that $p(e) = y$. This maps homeomorphically to $p^{-1}[Y]$ by projection onto the second factor, and an explicit inverse is given by the map sending e to $(p(e), e)$. Both of these maps are compatible with respect to the various projections onto Y.∎

4. Let $x \in X$ and let U be an evenly covered open neighborhood of x. Since X is totally disconnected there is an open subneighborhood $V \subset U$ such that V is open and closed in X. By continuity $p^{-1}[V]$ is open and closed in E.

If $y \in Y$, let $p(y) = x$, so that x has an evenly covered open neighborhood V which is also closed in X. Now let W be the sheet over V which contains y. We claim that W is open and closed in Y. Openness follows because W is a sheet over an evenly covered open subset of X. To prove that W is closed, let W' be the union of all the other sheets over V, so that W' is open and $E - W'$ is closed. Then $W = p^{-1}[V] \cap (X - W')$ shows that W is an intersection of two closed subsets and hence W is closed in Y.∎

5. (i) More generally, if $\mathcal{B}$ is a base for the topology of X and $E \to X$ is a covering space projection, then the subset $\mathcal{B}'$ of all evenly covered open subsets of $\mathcal{B}$ is also a base because every evenly covered open set is also a union of basic open subsets. Therefore if X is second countable, then there is a countable base $\mathcal{B}$ of X by evenly covered open subsets.

Let $\mathcal{A}$ be the family of all open subsets $W \subset E$ which are sheets over evenly covered open subsets $V \subset X$ such that $V \in \mathcal{B}$. We claim that $\mathcal{A}$ is a base for the topology on E. Since the number of sheets over each V is countable by the countability assumption and the number of open sets in $\mathcal{B}$ is countable, it follows that $\mathcal{A}$ is a countable family. Therefore we need only show that $\mathcal{A}$ is a base for the topology.

Let $\mathcal{C}$ be the family of open subsets in E such that $W \in \mathcal{C}$ if and only if W is a sheet over an evenly covered open subset $V \subset X$. Then $\mathcal{C}$ is a base for the topology on E, so it will suffice to show that every subset in $\mathcal{C}$ is a union of open subsets in $\mathcal{A}$. Let $W \in \mathcal{C}$ as above, and assume it is a sheet over V. Then we can write $V = \cup_j U_j$ as a countable union of open sets $U_j \in \mathcal{B}$, and it follows that W is the union of the open subsets $N_j = p^{-1}[U_j] \cap W$. By definition the sets N_j belong to $\mathcal{A}$, so we have shown that $\mathcal{A}$ forms a base for the topology of E.■

(*ii*) Follow the hint. By a previously proved exercise in Munkres we know that E is $\mathbf{T}_3$, and by the first part of this exercise it is also second countable. Therefore the Urysohn Metrization Theorem implies that E is metrizable.■

VIII.5 : Simply connected spaces

Additional exercises

1. (*i*) We need to show that $g \cdot x = x$ for some x only if $g = 1$. Let (u, v) denote a point of $S^2 \times S^3$, and suppose that $g \cdot (u, v) = (u, v)$; *i.e.*, $g \cdot u = u$ and $g \cdot v = v$. If $g \in C_n$ then $g \cdot (u, v) = (u, g \cdot v)$, and the second coordinate is equal to v if and only if $g = 1$. On the other hand, if $g \notin C_n$ then $g \cdot (u, v) = (-u, v) \neq (u, v)$, so that $g \cdot (u, v) = (u, v)$ implies that g must be equal to 1.

(*ii*) The preceding result yields a free action of $D(2n)$ on the simply connected space $S^2 \times S^3$, and by the results in this section of the notes the fundamental group of the quotient space $X_n = S^2 \times S^3 / D(2n)$ has a fundamental group isomorphic to $D(2n)$.■

(*iii*) Let $\Delta : D(2n) \to \mathbb{Z}_2$ be the homomorphism defined in the exercise, and identify $\mathbb{Z}_2$ with ± 1 as usual. Then the coordinate projection $p : S^2 \times S^3 \to S^2$ has the property that $p(g \cdot (u, v)\,) = (\Delta(g) \cdot u)$, and this implies that p passes to a map q from X_n to $\mathbb{RP}^2$. Let γ be a great circle curve in S^2 which joins a basepoint z to its antipodal point $-z$, let $v_0 \in S^3$, take $\beta(t) = (\gamma(t), v_0) \in S^2 \times S^3$, and let $\rho : S^2 \times S^3 \to X$ be the quotient projection. Then $\rho \circ \gamma$ is a closed durve in X, and its image $q \circ \rho \circ \gamma$ generates the fundamental group of $\mathbb{RP}^2$. Therefore the map from $\pi_1(X)$ to $\pi_1()$ is nontrivial. Further analysis would show that the kernel of this map in fundamental groups is just C_n, but this was not asked for in the exercise.■

2. If the inclusion $S^1 \subset \mathbb{RP}^n$ were a retract, then it would induce a monomorphism of fundamental groups. Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(\mathbb{RP}^n)$ is finite for $n \geq 2$, this cannot happen.■

3. (*i*) Under the given conditions, the result from this section imply that the map $p_*; \pi_1(X) \to \pi_1(\mathbb{RP}^n \times \mathbb{RP}^n) \cong \mathbb{Z}^2 \times \mathbb{Z}_2$ is injective and the number of sheets in the covering is the index of the image of p_*. If there is only one sheet, then this map must be a homeomorphism (it is 1–1, onto, continuous and open). Otherwise the number of sheets is the index of a subgroup of a group of order 4, and as such this number must be finite and even.■(*ii*) If E is a connected space covering space of X satisfying the default hypotheses and $\pi_1(X)$ is finite of odd order, then $\pi_1(E)$ is isomorphic to a subgroup of $\pi_1(X)$ as in the first part of this exercise, but now the index of the subgroup must divide the odd number $|\pi_1(X)|$ and hence the index, which equals the number of sheets in the covering, must be (finite and) odd. In particular, the index cannot be equal to 2.■

VIII.6 : Homotopy of paths and line integrals

Additional exercises

1. Recall the definition of the winding number integral: If $p(z)$ is never zero on a circle C_R of radius R about the origin and $\Gamma(p, R)$ be the closed curve given by $p(\exp(R \cdot 2\pi\, i\, t))$ for $0 \leq t \leq 1$, then the winding number integral is equal to

$$\int_{\Gamma(p,R)} \frac{x\,dy - y\,dx}{x^2 + y^2}$$

and its value is an integral multiple of 2π. By the results of this section, the integral factor is equal to the degree of the following composite

$$S^1 \xrightarrow{R\times} C_R \xrightarrow{p} \mathbb{C} - \{0\} \xrightarrow{\mathbf{u}} S^1$$

where $R\times$ denotes multiplication by R and $\mathbf{u}$ sends z to $|z|^{-1} \cdot z$.

The preceding sentence implies that if the winding number is not zero then the displayed map is not homotopic to a constant in $\mathbb{C} - \{0\}$ and hence that $\Gamma(p, R)$ does not extend to a continuous map from D^2 into $\mathbb{C} - \{0\}$. Now $p(r \cdot \mathbf{v})$ is a continuous extension of $\Gamma(p, R)$ to a map $D^2 \to \mathbb{C}$, and by the preceding sentence we know that its image cannot be contained in $\mathbb{C} - \{0\}$. Therefore its image must contain the point 0; in other words there must be some z_0 such that $|z| < R$ and $p(z_0) = 0$.■

2. The underlying idea is to show that the winding number is defined for $p + q$ and it is equal to the winding number of p, which by hypothesis is nonzero.

First of all, the condition $|q| < |p|$ for $|z| \leq R$ implies that $|p + q| \geq |p| - |q| > 0$ for $|z| = R$, and therefore the winding number of $p+q$ can be defined. To prove the winding numbers are equal, it is only necessary to show that $\Gamma(p, R)$ and $\Gamma(p + q, R)$ are homotopic as maps into $\mathbb{C} - \{0\}$. One obvious idea is to consider the straight line homotopy $p + t\,q$ where $0 \leq t \leq 1$ and show that its image lies in $\mathbb{C} = \{0\}$. The verification of this statement is a slight embellishment on what was already shown: $|p + tq| \geq |p| - t|q| \geq |p| - |q| > 0$.■

3. We know that the value of the integral only depends upon the free homotopy class of γ, and since there are only countably many free homotopy classes in $[S^1, U]$ it follows that there are only countably many possible values for the integral. Furthermore, since the line integral of a concatenated curve is the sum of the line integrals of the two pieces, it follows that the line integral of f defines a homomorphism $S(f)$ from $\pi_1(U)$ to the additive complex numbers, with the value of $S(f)$ only depending upon the image of a class in $\pi_1(U)$ in $[S^1, U]$.

If the line integral has only finitely many values, then $S(f)$ has a finite image, and since $S(f)$ is a homomorphism it follows that every element in the image has finite order. However, the only elment of finite additive order in $\mathbb{C}$ is 0, and therefore it follows that the image of $S(f)$ must be $\{0\}$ and there is only one possible value for the line integral.■