SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 9

Fall 2014

IX. Computing fundamental groups

IX.1 : Free groups

Additional exercises

1. For the sake of definiteness, assume that x and y are free generators for F. If H has index 2 in F, then H is a normal subgroup and $F/H \cong \mathbb{Z}_2$. Therefore every such subgroup is the kernel of a surjection from F to $\mathbb{Z}_2$. CLAIM 1: This is a 1–1 correspondence.

We can check this directly. There are three nontrivial homomorphisms from F into $\mathbb{Z}_2$, and they are completely determined by the values of a homomorphism on the generators. Consider the effect of each example on the set $S = \{x, xy, y\}$. For the homomorphism sending (x, y) to $(1, 0)$ the intersection of S with the kernel is $\{x\}$, for the homomorphism sending (x, y) to $(0, 1)$ the the intersection of S with the kernel is $\{y\}$, and for the homomorphism sending (x, y) to $(1, 1)$ the the intersection of S with the kernel is empty.

Therefore there are exactly three subgroups of index 2 in F, one of which is normally generated by x and y^2, another of which is normally generated by y and x^2, and the last of which is normally generated by $x^{-1}y$ and x^2 (or $y^{-1}x$ and x^2, or xy and x^2, *etc.*).■

2. We shall verify that F/H satisfies the appropriate Universal Mapping Property with free generators corresponding to the elements of $X - Y$. Let G be a group, let $j : X \to F$ identify X with a subset of F, and let $h : X - Y \to G$ be a map of sets. Extend h to a mapping $h_0 : X \to Y$ by setting $h(y) = 1$ if $y \in Y$. Since F is free on X, there is a homomorphism $\eta_0 : F \to G$ such that $\eta_0 \circ j(x) = h_0(x)$ for all $x \in X$. Since $y \in Y$ implies $h_0(y) = 1$, the kernel of η_0 contains the normal subgroup H, and if $\pi : F \to F/H$ is the quotient map then there is a unique homomorphism $\eta : F/H \to G$ such that $\eta \circ \pi = \eta_0$.

Before proving that F/H has the Universal Mapping Property for $X - Y$, we need to check that the map $\pi \circ j : X \to F/H$ is 1–1. Perhaps the easiest way to do this is to let A be the free group on $X - Y$ and take the homomorphism $F \to A$ which sends $X - Y$ to these free generators and sends Y to $\{1\}$. Since the map $X - Y \to F \to F/H \to A$ is 1–1, it follows that $X - Y \to F/H$ is also 1–1, confirming what we expected.

To complete the proof of the Universal Mapping Property, identify $X - Y$ with a subset of F/H via the map sending x to $j'(x) = \pi \circ j(x)$, so that $\eta \circ j'(x) = \eta_0 \circ j(x) = h_0(x)$ for all $x \in X - Y$. Therefore F/H has the universal mapping property for $X - Y$.■

3. (i) Let $F'_n \subset F_n$ be the commutator subgroup; then the quotient is isomorphic to the free abelian group $A_n \cong \mathbb{Z}^n$ on n generators. If $T : F_n \to F_n$ is an automorphism and $h : F_n \to A_n$ is the quotient projection as in the statement of the exercise, then the kernel of $h \circ T$ must contain the commutator subgroup $F'_n = [F_n, F_n]$ because the image of the homomorphism is abelian. Therefore there is a unique homomorphism $\theta(T) : A_n \to A_n$ such that $h \circ T = \theta(T) \circ h$.

Suppose now that we are given two automorphisms T_1 and T_2. By the uniqueness statement proved in the preceding paragraph, it is enough to show that $h \circ T_1 \circ T_2 = \theta(T_1) \circ \theta(T_2) \circ h$. This follows because $h \circ T_1 \circ T_2 = \alpha(T_1) \circ h \circ T_2 = \alpha(T_1) \circ \alpha(T_2) \circ h$.■

(*ii*) The statement in the hint is true because $h \circ \mathrm{id}_{F_n} = h = \mathrm{id}_{A_n} \circ h$. Therefore if $S = T^{-1}$ we have

$$\mathrm{id}_{A_n} \;=\; \theta(\mathrm{id}_{F_n}) \;=\; \theta(S \circ T) \;=\; \theta(S) \circ \theta(T)$$

and by interchanging the roles of S and T we also have $\mathrm{id}_{A_n} = \theta(T) \circ \theta(S)$. Therefore $\theta(T)$ is an automorphism, and predictably its inverse is $\theta(S)$.■

(*iii*) Again for definiteness, let x and y denote the generators of F_2 which project down to the elements $(1,0)$ and $(0,1)$ in $A_2 \cong \mathbb{Z}^2$. Following the hint, we shall find automorphisms of F_2 which induce θ on A_2 for choices of θ corresponding to each one of the three given generators. To avoid space-consuming displays of 2×2 matrices we shall refer to the displayed matrices, in order from left to right, as the *diagonal generator*, the *transposition generator*, and the *shear generator*. For the diagonal generator, take the self-homomorphism T of F_2 which sends x to x^{-1} and y to itself; such a homomorphism exists because F_2 is free, and it is an automorphism because $T \circ T = \mathrm{id}$ (it is only necessary to check this on the free generators), so that T is equal to its own inverse. For the transposition generator, take the self-homomorphism T which interchanges x and y; once again $T \circ T = \mathrm{id}$ implies that T is its own inverse. Finally, for the shear generator, take the homomorphism T sending x to itself and y to xy. For this example we claim that the inverse is the homomorphism sending x to itself and y to $x^{-1}y$. Once again, to prove that $S \circ T$ and $T \circ S$ are the identities, it is enough to do so on the standard set of free generators. Clearly we have $S \circ T(x) = x = T \circ S(x)$ since $S(x) = T(x) = x$, and we also have

$$S \circ T(y) \;=\; S(xy) \;=\; S(x)S(y) \;=\; x \cdot (x^{-1}y) \;=\; y$$

$$T \circ S(y) \;=\; T(x^{-1}y) \;=\; T(x^{-1})T(y) \;=\; x^{-1} \cdot (xy) \;=\; y$$

and therefore we know that $S = T^{-1}$.■

4. (*i*) Take the map from F_{n-1} to G with sends the free generator $x_i \in F_{n-1}$ to $g_i \in G$. The extension of this map to a homomorphism is onto, and therefore G is isomorphic to a quotient of F_{n-1}.■

(*ii*) In any group G, if $g = g^1$ then either $g = 1$ or else $g^2 = 1$. The latter cannot happen in an odd order group unless $G = 1$, so this means that the nontrivial elements of G can be decomposed into $\frac{1}{2}(|G| - 1)$ pairs of the form $\{g_i, h_i = g_i^{-1}\}$, where $1 \leq i \leq k$ and $|G| = 2k + 1$.

In this case take the map from F_k to G with sends the free generator $x_i \in F_k$ to $g_i \in G$. The extension of this map to a homomorphism is onto, and therefore G is isomorphic to a quotient of F_k.■

IX.2 : Sums and pushouts of groups

Problems from Munkres, § 68, p. 421

2. (*a*) Let $1 \neq x_i \in G_i$ for $i = 1, 2$; then $x_1 x_2 x_1^{-1} x_2^{-1}$ is a reduced word, and by Step 4 in the proof of Munkres, Theorem 68.2 we know that this element is not the identity in $G_1 * G_2$. But this means that $x_1 x_2 \neq x_2 x_1$ whenever x_1 and x_2 are nontrivial elements of G_1 and G_2 respectively.■

(b) If x is a reduced word of even length, write it in the form $a_1 b_1 \cdots a_k b_k$ where each a_j lies in one of the groups G_i and each b_j lies in the other group. It follows that for each $n > 0$ that x^n corresponds to the reduced word $a_1 b_1 \cdots a_{nk} b_{nk}$ where the sequences satisfy the periodicity conditions $a_j = a_{j+k}$ and $b_j = b_{j+k}$ for $j \le nk - k$. Since this is also a nontrivial word, it follows from the same reasoning as before that $x^n \neq 1$ in the free product. Therefore x has infinite order.

Suppose now that we have a reduced word x of odd length ≥ 3 (*this was* not *part of the problem as stated in Munkres, but clearly it is indispensable because a reduced word of length* 1 *cannot be conjugate to anything shorter*). In analogy with the preceding paragraph, write x in the form $b_0 a_1 b_1 \cdots a_k b_k$ where a_j and b_j are as before. We can easily find a shorter word which is conjugate to the given one because $b_0^{-1} x b_0$ is equal to $a_1 b_1 \cdots a_k (b_k b_0)$. There are now two possibilities. If $b_k b_0 \neq 1$, then we have shown that x is conjugate to an element corresponding to a reduced word of even length $2k$. If $b_k b_0 = 1$, then we have shown that x is conjugate to an element corresponding to a reduced word of odd length $2k - 1$.■

(c) By part (b) and induction, every nontrivial word is either conjugate to a word whose length is either an even number or 1 (look at the shortest word in the conjugacy class, and note that a nontrivial word cannot be conjugate to the empty word). If the word x is conjugate to a word y of even length, then the orders of x and y are equal, and since y has infinite order it follows that the same holds for x. On the other hand, if x is conjugate to a word y of length 1, we know that y must correspond to a nontrivial element of G_1 or G_2, and if x has finite order then y must also have the same finite order.■

3. The easiest way to solve this exercise might be to look at the images of everything in the direct product $G_1 \times G_2$. The Universal Mapping Property for free products guarantees the existence of a homomorphism $\theta : G_1 * G_2 \to G_1 \times G_2$ such that $\theta \circ i_1(a) = (a, 1)$ and $\theta \circ i_2(b) = (1, b)$, where i_t denotes the standard injection of G_t into $G_1 \cap G_2$. The problem does not require a proof that cG_1c^{-1} is a subgroup, but this follows quickly from the fact that the latter is the image of G_1 under the conjugation automorphism of $G_1 * G_2$ sending x to cxc^{-1}.

Suppose that $a \in G_1$ is such that $cac^{-1} \in G_2$ It then follows that $\theta(a) \in G_1 \times \{1\}$ and $\theta(c)\theta(a)\theta(c)^{-1} \in \{1\} \times G_2$. CLAIM: $\theta(c)\theta(a)\theta(c)^{-1} \in G_1 \times \{1\}$, and this element corresponds to a conjugate of a in G_1. — If this is true, then $\theta(c)\theta(a)\theta(c)^{-1}$ belongs to $(G_1 \times \{1\}) \cap (\{1\} \times G_2)$, which is the trivial group, and furthermore a is conjugate to this element in G_1. In particular, a is conjugate in G_1 to the trivial element, and this implies that $a = 1$. To summarize, the claim implies that if $cac^{-1} \in G_2$ then $a = 1$ and therefore also $cac^{-1} = 1$.

To prove the assertions regarding $\theta(c)\theta(a)\theta(c)^{-1}$, write $c = u_1 v_1 \cdots u_k v_k$ where $u_j \in G_1 \times \{1\}$ and $v_j \in \{1\} \times G_2$. If $c \neq 1$ we can do this using either a reduced word of even length or taking a reduced word of odd length and setting $u_1 = 1$ (if the word starts and ends with something from G_2) or $v_k = 1$ (if the word starts and ends with something from G_1). Since the images of G_1 and G_2 commute with each other, an inductive argument shows that

$$\theta(c)\,\theta(a)\theta(c)^{-1} \quad = \quad \theta(u_1 v_1 \cdots u_k v_k)\,\theta(a)\theta(u_1 v_1 \cdots u_k v_k)^{-1} \quad =$$

$$\theta(u_1 v_1 \cdots u_{k-1} v_{k-1})\,\theta(u_k a u_k^{-1})\theta(u_1 v_1 \cdots u_{k-1} v_{k-1})^{-1} \quad = \quad \cdots \quad = \quad \theta(u_1 \cdots u_k\, a\, u_k^{-1} \cdots u_k^{-1})$$

where the expression in the last term is an element of G_1 which is conjugate (in G_1) to a. This is the claim in the preceding paragraph.■

1. To simplify the notation, if H is a group, then $\mathbf{Ab}(H)$ will denote the quotient $H/[H,H]$, where $[H,H]$ is the commutator subgroup (which is normal in H). We shall also denote $G_1 * G_2$ by G as in the statement of the exercise.

Starting with the abelinization homomorphisms $\alpha_i : G_i \to \mathbf{Ab}(G_i)$, we can define a homomorphism $\theta : G \to \mathbf{Ab}(G_1) \oplus \mathbf{Ab}(G_2)$ whose restriction to $G_1 \subset G$ is the map sending a to $(a,0)$ and whose restriction to $G_2 \subset G$ is the map sending b to $(0,b)$. By construction θ is onto, and since the codomain is an abelian group the kernel of θ must contain the commutator subgroup. Therefore θ factors as a composite $G \to \mathbf{Ab}(G) \to \mathbf{Ab}(G_1) \oplus \mathbf{Ab}(G_2)$, where the first arrow is abelianization and the second will be denoted by φ.

For the same general reasons, the composites $G_i \to G = G_1 * G_2 \to \mathbf{Ab}(G)$ have factorizations $G_i \to \mathbf{Ab}(G_i) \to \mathbf{Ab}(G)$, and the induced maps of abelianizations will be denoted by J_i. Therefore we can define a homomorphism

$$\psi : \mathbf{Ab}(G_1) \oplus \mathbf{Ab}(G_2) \;\longrightarrow\; \mathbf{Ab}(G)$$

such that $\psi(u,v) = J_1(u) + J_2(v)$. By construction the composites

$$G_i \to \mathbf{Ab}(G_i) \to \mathbf{Ab}(G) \to \mathbf{Ab}(G_i) \quad = \quad G_i \to G \to G_i \to \mathbf{Ab}(G_i)$$

are the abelianization mappings, and therefore the composites $\mathbf{Ab}(G_i) \to \mathbf{Ab}(G) \to \mathbf{Ab}(G_i)$ are identity mappings. Similarly, if $i \neq j$ then the triviality of the composites $G_i \to G \to G_j$ implies that the abelianized mappings $\mathbf{Ab}(G_i) \to \mathbf{Ab}(G) \to \mathbf{Ab}(G_j)$ are zero homomorphisms. If we combine these with the definitions of φ and ψ, we see that $\varphi \circ \psi$ is the identity on $\mathbf{Ab}(G_1) \oplus \mathbf{Ab}(G_2)$. We claim these maps are isomorphisms, and to prove this it will suffice to show that ψ is onto. However, this follows quickly because we know that G is generated by the images of G_1 and G_2, which implies that $\mathbf{Ab}(G)$ is generated by the images of $\mathbf{Ab}(G_1)$ and $\mathbf{Ab}(G_2)$.■

3. For the sake of definiteness, we shall assume that $m \geq n$ (it will be clear that the case $m \leq n$ can be handled similarly).

(a) If G_1 and G_2 are abelian groups, then Exercise 1 implies that $\mathbf{Ab}(G_1 * G_2) \cong G_1 \oplus G_2$. If we specialize to the case where $G_1 = \mathbb{Z}_m$ and $G_2 = \mathbb{Z}_n$, this implies that $\mathbf{Ab}(G_1 * G_2)$ is a finite group of order mn.■

(b) Follow the hint. By Exercise 68.2 in Munkres, the only elements of finite order in $G_1 * G_2$ are those which are conjugate to elements in either G_1 or G_2, and thus if $g \in \mathbb{Z}_m * \mathbb{Z}_n$ has finite order, this order must divide either m or n. Since we are assuming that $m \geq n$, the largest possible order is m, and in fact this order is realized by the generator of $\mathbb{Z}_m$.■

(c) If $G = \mathbb{Z}_m * \mathbb{Z}_n$, then $|\mathbf{Ab}(G)| = mn$ implies that mn is uniquely determined by G, and by (b) we know that m is uniquely determined by G. Therefore $n = (mn)/m$ is also uniquely determined by G.

4. The goal of the problem is to find finite abelian groups G_1, G_2, H_1, H_2 such that $|G_1| \neq |H_1|$ and $|G_2| \neq |H_2|$ such that $G_1 \times G_2 \cong H_1 \times H_2$, and the hint is to use the abstract version of the Chinese Remainder Theorem: $\mathbb{Z}_a \times \mathbb{Z}_b \cong \mathbb{Z}_{ab}$ *if a and b are relatively prime.* — The latter can be found in nearly every upper level undergraduate textbook on abstract algebra or elementary number theory, so we shall not prove it here.

Since $2, 3, 5$ are pairwise relatively prime the Chinese Remainder Theorem and $(2 \cdot 3) \cdot 5 = 30 = 2 \cdot (3 \cdot 5)$ imply that

$$\mathbb{Z}_{30} \cong \mathbb{Z}_6 \times \mathbb{Z}_5 \ , \qquad \mathbb{Z}_{30} \cong \mathbb{Z}_2 \times \mathbb{Z}_{15}$$

so we get the desired examples if we take $G_1 = \mathbb{Z}_6$, $G_2 = \mathbb{Z}_5$, $H_1 = \mathbb{Z}_2$, and $H_2 = \mathbb{Z}_{15}$.■

Additional exercises

1. We shall repeatedly use the fact that a free product of two groups $*_i \, L_i$ is uniquely characterized up to isomorphism by the fact that homomorphisms from $*_i \, L_i$ to another group M correspond bijectively to homomorphisms from the summands L_i into M, and the correspondence is given by restricting to the subgroups L_i.

For $(G * H) * K$, the preceding paragraph means that homomorphisms from this group to some other group M are in 1–1 correspondence with homomorphisms from $G * H$ and K into M. However, homomorphisms from $G * H$ into M in 1–1 correspondence with homomorphisms from G into M and from H into M. Combining these, we see that homomorphisms from $(G * H) * K$ into M are in 1–1 correspondence with homomorphisms $G \to M$, $H \to M$ and $K \to M$. Since this is the defining condition for a free product of the three groups G, H and K it follows that $(G * H) * K$ is in fact a free product of these three groups. Similarly, homomorphisms from $G * (H * K)$ into some other group M are in 1–1 correspondence with homomorphisms from G and $H * K$ into M, and since homomorphisms from $H * K$ into M in 1–1 correspondence with homomorphisms from H into M and from K into M, we see that homomorphisms from $G * (H * K)$ into M are in 1–1 correspondence with homomorphisms $G \to M$, $H \to M$ and $K \to M$; as before, this means that $G * (H * K)$ is in fact a free product of G, H and K.

Finally, homomorphisms from both $G * H$ and $H * G$ into an arbitrary group H correspond bijectively to homomorphisms $G \to M$ and $H \to M$, and this yields an isomorphism between $G * H$ and $H * G$.■

2. (i) Follow the hint and note that $K * K \cong K$ because $K * K$ is a free group on $\aleph_0 + \aleph_0 = \aleph_0$ generators if K is a free group on $\aleph_0$ generators. If H_1 is finite but nontrivial and $H_2 = H_1 * K$, then H_1 is finite but H_2 is infinite. On the other hand, we have

$$H_2 * K \cong (H_1 * K) * K \cong H_1 * (K * K) \cong H_1 * K$$

which is what we wanted to prove.■

(ii) The underlying ideas are the same, but here we have $K \times K = K \oplus K$ is isomorphic to K because $K \oplus K$ is a free abelian group on $\aleph_0$ generators if K is. If H_1 is finite but nontrivial and $H_2 = H_1 * \times K$, then H_1 is finite but H_2 is infinite. On the other hand, we have

$$H_2 \times K \cong (H_1 \times K) \times K \cong H_1 \times (K \times K) \cong H_1 \times K$$

which is what we wanted to prove.■

(iii) Once again, this is the same basic idea, but now we are working with topological spaces. An explicit isometry from $Y \times Y$ to Y is given as follows: Let $\{\mathbf{e}_j\}$ denote the set of standard unit vectors in Y, and take the linear isomorphism from $Y \times Y$ to Y which sends $(\mathbf{e}_j, \mathbf{0})$ to $\mathbf{e}_{2j-1}$ and $(\mathbf{0}, \mathbf{e}_j)$ to $\mathbf{e}_{2j-1}$. This is clealy an invertible linear transformation, and if one imposes the metrics associated to the usual dot products (so that the unit vectors are orthonormal and $Y \times \{\mathbf{0}\}$ is orthogonal to $\{0\} \times Y$), then this linear isomorphism is an isometry of inner product spaces, which implies among other things that $Y \times Y$ is homeomorphic to Y.

Let X_1 be a compact metric space, and let $X_2 = X_1 \times Y$. Then X_1 is not homeomorphic to X_2 but $X_1 \times Y$ is homeomorphic to $X_2 \times Y$ because we have $X_2 \times Y = X_1 \times Y \times Y \cong X_1 \times Y$.■

3. As before the proof that $h \times h$ is the identity reduces to showing that $h \times h$ maps the free generators x and y to them selves. Since $h(x) = y$ and $h(y) = x$, this follows immediately.

Suppose now that $h(w) = w$ for some nontrivial reduced word w in x and y. If w begins with a power of x then $h(w)$ begins with a power of y and vice versa. By the unique factorization property for nontrivial reduced words, it follows that $h(w)$ cannot be equal to w.■

Note. In contrast, the induced automorphism $\theta(h)$ of $\mathbb{Z}^2$ sends $(1,1)$ to itself.

IX.3 : The Seifert – van Kampen Theorem

Problems from Munkres, § 70, *p.* 433

1. By the Seifert-van Kampen Theorem it will suffice to show that

$$\begin{array}{ccc}
\pi_1(U \cap V, p) & \xrightarrow{i_{1*}} & \pi_1(U,p) \\
\downarrow i_{2*} & & \downarrow \\
\pi_1(V,p) & \longrightarrow & \pi_1(U,p)/N_1 \, * \, \pi_1(V,p)/N_2
\end{array}$$

is a pushout diagram if i_{1*} and i_{2*} are trivial homomorphisms, where the maps from $\pi_1(U,p)$ and $\pi_1(V,p)$ to the respective quotients $\pi_1(U,p)/N_1$ and $\pi_1(V,p)/N_2$ followed by the usual injections J_1 and J_2 into the free product. Denote the quotient projections $\pi_1(U,p) \to \pi_1(U,p)/N_1$ and $\pi_1(V,p) \to \pi_1(V,p)/N_2$ by q_1 and q_2 respectively.

Suppose that we are given homomorphisms $A : \pi_1(U,p) \to G$ and $B : \pi_1(V,p) \to G$ such that $A \circ i_{1*} = B \circ i_{2*}$. Since i_{1*} and i_{2*} are trivial it follows that we have factorizations through the respective quotients; *i.e.*, we have $A = A' \circ q_1$ and $B = B' \circ q_2$ for uniquely determined homomorphisms A' and B'. By the Universal Mapping Property for free products, there is a unique homomorphism C from the free product into G whose restrictions to $\circ J_1$ and $C \circ J_2$ to $\pi_1(U,p)/N_1$ and $\pi_1(V,p)/N_2$ are equal to A' and B' respectively, and therefore we also have $C \circ (J_1 \circ q_1) = A' \circ q_1 = A$ and $C \circ (J_2 \circ q_2) = B' \circ q_2 = B$. To complete the proof, we need to show that if $D : \pi_1(U,p)/N_1 * \pi_1(V,p)/N_2$ is an arbitrary homomorphism such that $D \circ (J_1 \circ q_1) = A$ and $D \circ (J_2 \circ q_2) B$, then $D = C$. If D satisfies these conditions then we have $D \circ J_1 \circ q_1 = C \circ J_1 \circ q_1$ and $D \circ J_2 \circ q_2 = C \circ J_2 \circ q_2$; since q_1 and q_2 are onto, the given equations imply that $D \circ J_1 = C \circ J_1$ and $D \circ J_2 = C \circ J_2$. We can now use the uniqueness condition in the Universal Mapping Property for free products to conclude that $D = C$. This completes the proof that the diagram at the beginning of this solution is a pushout.■

3. (a) If G_1 has a finite generator set X_1 with a finite relation set R_1 and G_2 has a finite generator set X_2 with a finite relation set R_2, then $G_1 * G_2$ has a finite generator set $X_1 \amalg X_2$ with a finite relation set $R_1 \amalg R_2$,■

(b) Follow the hint, but work more generally with a pushout

$$\begin{array}{ccc}
K & \xrightarrow{i_1} & H_1 \\
\downarrow i_2 & & \downarrow j_1 \\
G_2 & \xrightarrow{j_2} & G
\end{array}$$

where K is finitely generated and H_1 and H_2 are finitely presented.

The construction of pushouts in Section IX.2 shows that G is isomorphic to the quotient of $\Gamma = G_1 * G_2$ by the normal subgroup N which is generated by all elements of the form $i_1^{-1}(k)\, i_2(k)$, where k runs through all the elements of G. CLAIM: *If $k_1, \cdots, k_r$ generate K, then N is also the smallest normal subgroup containing the finite set $S = \{i_1^{-1}(k_t)\, i_2(k_t) \mid 1 \leq t \leq r\}$.*

If the claim is true, we can complete the solution as follows: By Exercise 1 we know that $G_1 * G_2$ is finitely presented, and by the claim we know that N is finitely normally generated, so if $G_1 * G_2$ is presented with finite generating set X and finite relation set R, then we obtain the quotient by expanding R to a set which also includes a finite family of words in the generators which map to the elements in the set S.

We now prove the claim. Let N_0 be the subgroup normally generated by S, so that $N_0 \subset N$. To prove the reverse inclusion, consider the map of quotient groups $\pi : \Gamma/N_0 \to \Gamma/N$ which sends each coset of N_0 to the coset N which contains it, and let $\rho : G \to G/N_0$ be the usual quotient space projection. By definition, N_0 is normally generated by the elements $i_1^{-1}(k_t)\, i_2(k_t)$, and therefore $\rho \circ i_1(k_t) = \rho \circ i_2(k_t)$ for all t. Since ρ, i_1 and i_2 are homomorphisms and the elements k_t generate K, it follows that $\rho \circ i_1(k) = \rho \circ i_2(k)$ for all $k \in K$. But this means that the normal subgroup N_0 contains all elements of the form $i_1^{-1}(k)\, i_2(k)$ where $k \in K$, and since these elements normally generate N it follows that all of N_0 is contained in N.■

Additional exercises

1. (i) For this part of the exercise, in the pushout diagram

$$\begin{array}{ccc} \pi_1(U \cap V, p) & \xrightarrow{i_{1*}} & \pi_1(U, p) \\ \downarrow i_{2*} & & \downarrow j_{1*} \\ \pi_1(V, p) & \xrightarrow{j_{2*}} & \pi_1(X, p) \end{array}$$

we know that $\pi_1(U, p)$ is trivial and $\pi_1(V, p)$ is abelian. It will suffice to prove that the map $\pi_1(V, p) \to \pi_1(X, p)$ is onto. We know that $\pi_1(X, p)$ is generated by the images of $\pi_1(U, p)$ and $\pi_1(V, p)$ and since the image of the first group must be trivial it follows that $\pi_1(V, p)$ generates $\pi_1(X, p)$, which means that $\pi_1(V, p) \to \pi_1(X, p)$ is onto.■

(ii) Let X be the Figure Eight Space which is a union of two closed subspaces $C_1 \cup C_2$ such that each is homeomorphic to S^1 and $C_1 \cap C_2$ consists only of the basepoint p. Choose points $q_i \in C_i - \{p\}$, and let U_1 and U_2 be $X - \{q_2\}$ and $X - \{q_1\}$ respectively (note the switch in subscripts — this is not a misprint). Then C_i is a strong deformation retract of U_i and $U_1 \cap U_2$ is contractible, so that the pushout diagram associated to $\pi_1(X = U_1 \cup U_2$ is given as follows:

$$\begin{array}{ccc} \{0\} \cong \pi_1(U \cap V, p) & \xrightarrow{i_{1*}} & \pi_1(U, p) \cong \mathbb{Z} \\ \downarrow i_{2*} & & \downarrow j_{1*} \\ \mathbb{Z} \cong \pi_1(V, p) & \xrightarrow{j_{2*}} & \pi_1(X, p) \cong F_2 \end{array}$$

In this example the fundamental groups of U and V are abelian but the fundamental group of X is not.■

2. (*i*) Once again we know that $\pi_1(X)$ is generated by the images of $\pi_1(U)$ and $\pi_1(V)$. Since $\pi_1(U \cap V)$ maps onto both of the latter groups, it follows that all the generators for $\pi_1(X)$ actually lift back to $\pi_1(U \cap V)$.■

(*ii*) As in (*i*) if A and B denote generating sets for $\pi_1(U)$ and $\pi_1(V)$ respectively and A' and B' denote their images in $\pi_1(X)$, then $A' \cup B'$ generates $\pi_1(X)$. But if A and B are finite, then so is $A' \cup B'$.■

3. We begin with a general statement. Suppose that we have a group G presented as a quotient F/N where F is freely generated by X and N is normally generated by relations $R \subset F$. Then $\mathbf{Ab}(G)$ is isomorphic to $F/N\,[F,F]$, which is isomorphic to

$$(F/[F,F])/(N\,[F,F]/[F,F])$$

in which $N\,[F,F]/[F,F]$ is the image of $N \subset F \to F/[F,F]$.

For the example in this exercise, the preceding observation shows that the abelianization $\mathbf{Ab}(G)$ is the quotient of $\mathbb{Z}^2$ modulo the subgroup generated by the single abelianized relation $(3,-2)$ because the abelianizations of the other two relations are trivial. One easy way of seeing that the quotient is infinite cyclic is to observe that the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ sending (x,y) to $2x+3y$ is onto and its kernel is the cyclic subgroup generated by $(3,-2)$.■

(*ii*) Let $\rho : G \to G/N$ be the quotient group projection. Since $xy^{-1} \in N$ it follows that $\rho(x) = \rho(y)$. By definition we know that $x^3 = y^2$, so that $\rho(x)^3 = \rho(y)^2$. If we combine the preceding two sentences we find that $\rho(x)^3 = \rho(x)^2$, which means that $\rho(x) = 1$ and hence also that $\rho(y) = \rho(x) = 1$; *i.e.*, we have $x, y \in N$. Since x and y generate G, this means that $N = G$.■

4. By the Seifert-van Kampen Theorem, it will suffice to prove the following algebraic result about pushout diagrams of groups: If we are given an onto homomorphism $i_1 : K \to H_1$ and an isomorphism $i_2 : K \to H_2$, then the following square is a pushout diagram:

$$\begin{tikzcd}
K \arrow[r, "i_1"] \arrow[d, "i_2"] & H_1 \arrow[d, "\mathrm{id}"] \\
H_1 \arrow[r, "i_1 i_2^{-1}"] & H_1
\end{tikzcd}$$

This square commutes because both composites from K to H_1 are equal to i_1.

As usual, we shall prove the given square is a pushout by verifying that it has the Universal Mapping Property. So let $f_1 : H_1 \to M$ and $f_2 : H_2 \to M$ satisfy $f_1 \circ i_1 = f_2 \circ i_2$. We need to find a unique map $h : H_1 \to M$ such that $h \circ j_i = f_i$, where j_1 is the identity and $j_2 = i_1 \circ i_2^{-1}$. If we take $h = f_1$, then $h \circ j_1 = h \circ \mathrm{id} = f_1$ and $h \circ j_2 = h \circ i_1 \circ i_2^{-1} = f_1 \circ i_2^{-1} = f_2 \circ i_2 \circ i_2^{-1} = f_2$, so there is a map from H_1 to M with the right properties. To show that a map with the right properties is unique, note that if $k \circ j_i = f_i$ for $i = 1, 2$ then $k = k \circ \mathrm{id} = k \circ j_1 = f_1$, so that $k = h$.■

5. Consider the associated pushout diagram:

$$\begin{tikzcd}
\mathbb{Z} \times \mathbb{Z} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathbb{Z} \arrow[d, "j_1"] \\
\mathbb{Z} \arrow[r, "j_2"] & A
\end{tikzcd}$$

Since p_1 and p_2 are onto, it follows that the composite $\mathbb{Z} \times \mathbb{Z} \to A$ is also onto (see Exercise 2); note that A is abelian because it is a homomorphic image of $\mathbb{Z} \times \mathbb{Z}$. Since $(1,0)$ and $(0,1)$ are in the kernels of p_2 and p_1 respectively, it follows that both these elements map to zero in A, and since the two elements in question generate $\mathbb{Z} \times \mathbb{Z}$, it follows that everything in $\mathbb{Z} \times \mathbb{Z}$ maps to zero in A. If we combine this with the conclusion of the previous paragraph, we see that A must be the trivial group.■

6. Follow the hint; let $U = M - Y$ and $V = M - X$. Then U is homeomorphic to $X \amalg [0, 1)$ with $x \in X$ identified to 0, and V is homeomorphic to $(0, 1] \amalg Y$ with 1 identified to $y \in Y$. Both U and V are open in M, and their intersection is homeomorphic to the open interval $(0, 1)$. Furthermore, X and Y are strong deformation retracts of U and V respectively. Therefore the Seifert-van Kampen Theorem implies that the fundamental group of M is the free product of the fundamental groups of X and Y (we have not been careful about the basepoints because the isomorphism type of the fundamental groups of the spaces in this exercise are isomorphic for all choices of basepoints).■

IX.4 : Examples and computations

Additional exercises

1. (*i*) One can also model X topologically as the subspace $C \subset \mathbb{R}^3$ given by $S^2 \cup \{(0.0)\} \times [-1, 1]$. We shall prove that this space is homeomorphic to the subspace of $\mathbb{R}^4$ described in the exercise as follows: Take the identity map on S^2, and map a point of the form $(0, 0, t)$, where $-1 \leq t \leq 1$, to the point $(0, 0, t, \sqrt{1 - t^2})$. One can check directly that this map is continuous and 1–1 onto, so it is a homeomorphism because X is compact Hausdorff.■

(*ii*) Follow the hint. The space which interests us is $S^2 \cup A$, and $D^3 \cup A$ is formed from it by regularly attaching a 3-cell, so by Proposition 3 IX.4.2 we know that $\pi_1(S^2 \cup A) \cong \pi_1(D^3 \cup A)$.

(*iii*) If $B \subset \mathbb{R}^3$ is the straight line segment described in the exercise which joins the north and south poles of S^2, then a retraction $D^3 \to B$ is defined by sending (x, y, z) to z; if we also take the straight line homotopy between these two points (which stays inside D^3 by convexity), we obtain deformation retract data for $B \subset D^3$. Now $A \cap B$ consists of the two points $\pm \mathbf{e}_3$, and by an exercise from Unit VII it follows that $B \cup A$ is a strong deformation retract of $D^3 \cup A$. Since $B \cup A$ is the union of two closed subspaces homeomorphic to $[-1, 1]$ which meet at their endpoints, the space $A \cup B$ is homeomorphic to S^1; for the sake of completeness, we note that an explicit homeomorphism is given by sending one copy of $[-1, 1]$ to the upper semicircle by the mapping $t \to (t, \sqrt{1 - t^2})$ and sending the other copy of $[-1, 1]$ to the lower semicircle by the mapping $t \to (t, -\sqrt{1 - t^2})$.

Finally, the preceding observations combine to yield the fundamental group relationships $\pi_1(S^2 \cup A) \cong \pi_1(D^3 \cup A) \cong \pi_1(B \cup A) \cong \pi_1(S^1) \cong \mathbb{Z}$, as asserted in the statement of the exercise.■

2. The intersection of $D^2 \times \{0\}$ with S^2 is equal to $S^1 \times \{0\}$, so if we take $A = S^2$ and $B = D^2$ then we have an example with the properties described in the discussion before the statement of Proposition IX.4.2. Therefore we can apply this result to conclude that the map of fundamental groups $\pi_1(S^2) \to \pi_1(X)$ is onto. Since $\pi_1(S^2)$ is trivial, it follows that $\pi_1(X)$ must also be trivial.■

3. (*i*) Follow the hint. The data in the problem yield the following commutative diagram, in which the vertical arrows j_k are isomorphisms:

$$\begin{array}{ccccccc}
\mathbb{Z} & \to \cdots & c_k^{-1}\mathbb{Z} & \xrightarrow{\subset} & c_{k+1}^{-1}\mathbb{Z} & \to \cdots & \mathbb{Q} \\
\downarrow j_1 & & \downarrow j_k & & \downarrow j_{k+1} & & \\
A_1 & \to \cdots & A_k & \xrightarrow{h_k} & A_{k+1} & \to \cdots & G
\end{array}$$

If we define $\varphi_k : c_k^{-1}\mathbb{Z} \to G$ by $g_k \circ j_k$, then we have the recursive property

$$\varphi_{k+1}|c_k^{-1}\mathbb{Z} \quad = \quad g_{k+1} \circ j_{k+1}|c_k^{-1}\mathbb{Z} \quad = \quad g_{k+1} \circ h_k \circ j_k \quad = \quad g_k \circ j_k \quad = \quad \varphi_k$$

and therefore we can assemble these mappings to produce a homomorphism $\varphi : \mathbb{Q} \to G$. This map is onto, for each $a \in G$ has the form $g_k(b)$ for some k and $b \in A_k$, so if $j_k(b') = b$ we have $\varphi_k(b') = a$. In particular, this implies that G is abelian, so we shall use 0 to denote the neutral element in G. To see that φ is 1–1, suppose that $x \in \mathbb{Q}$ maps to 0, and choose k such that $x \in c_k^{-1}\mathbb{Z}$. Then $j_k(x) \in A_k$ maps to 0 in G, and therefore there is some $M \geq 0$ such that $j_k(x)$ maps to 0 in A_{k+M}. But the map $A_k \to A_{k+M}$ is equivalent to a nonzero map $\mathbb{Z} \to \mathbb{Z}$, so if $j_k(x)$ maps to 0 in some A_{k+M} then we must have $j_k(x) = 0$. Since j_k is an isomorphism we must have $x = 0$. Therefore $\varphi : \mathbb{Q} \to G$ is an isomorphism.$\blacksquare$

(ii) The mapping $j[d]$ is 1–1 because the composite of $j[d]$ with projection onto the D^2 factor is the standard inclusion of S^1 in D^2, and the map $j[d]_*$ in fundamental groups corresponds to multiplication by d because the composite of $j[d]$ with projection onto the S^1 factor has degree d and this coordinate projection map induces an isomorphism $\pi_1(S^1 \times D^2) \to \pi_1(S^1)$.$\blacksquare$

(iii) A more concrete approach to constructing E is to describe it as a subspace of $\mathbb{R}^5 = \mathbb{C} \times \mathbb{C} \times \mathbb{R}$; more precisely, we shall realize each S_k as a subset of $\mathbb{C}^2 \times [k, k+1]$ such that the continuous mapping $S_k \to [k+1]$ corresponds to the last coordinate. Consider the subspace T_k of $\mathbb{C}^2 \times [k, k+1]$ consisting of $S^1 \times D^2 \times \{k+1\}$ together with the image of $S^1 \times [0,1]$ under the continuous mapping θ_k defined by

$$\theta_k(z,t) \quad = \quad (tz^{d_k} + (1-t)z,\, tz,\, t+k) \ .$$

We claim that θ_k is 1–1, and from this it follows that the standard quotient map from S_k to T_k is a homeomorphism onto its image. So suppose that $\theta_k(z,t) = \theta_k(z',t')$. Equating the third coordinates, we see that $t + k = t' + k$, so that $t = t'$. Now equating the second coordinates, we see that $tz = tz'$ so that either $t = 0$ (and hence $t' = 0$ or else $z = z'$; in the second case we are finished, so assume that $t = 0$ and look at the first coordinates. When $t = 0$ the first coordinate equation reduces to $z = z'$, so we have shown that $(z,t) = (z',t')$ must always hold. — Continuing, we see that the union $\cup_{j \leq k} T_k$ is homeomorphic to E_k, and if we set $T = \cup_k T_k$ we have a 1–1 onto continuous mapping $E \to T$. Projection onto the final coordinate in $\mathbb{R}^5 = \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ yields a continuous mapping from T to $[0,\infty)$ such that the composite $E \to T \to [0,\infty)$ has all the right properties. Furthermore, this mapping sends the inverse image of $[0,k)$ homeomorphically to the inverse image of $[0,k)$ for all k, and from this one can prove that the map $E \to T$ is actually a homeomorphism (but this will not be needed to carry out the computations).

We now need to verify the assertion about the maps in fundamental groups associated to the inclusions $E_k \to E_{k+1}$. To start, we claim that for each k the inclusion $S^1 \times \{0\} \times \{k+1\} \subset S_k$ is a deformation retract. Since we know that the inclusion $S^1 \times \{0\} \times \{k+1\} \subset S^1 \times D^2 \times \{k+1\}$ is a deformation retract, it will suffice to show that $S^1 \times D^2 \times \{k+1\} \subset S_k$ is a deformation retract. This follows because $S_k = F_1 \cup F_1$, where $F_1 = S^1 \times D^2 \times \{k+1\}$ and F_2 is homeomorphic to $S^1 \times [0,1]$ such that $S^1 \times \{1\}$ corresponds to $F_1 \cap F_2$. We can now proceed by induction on j to show that the inclusion

$$S^1 \times \{0\} \times \{k+1\} \quad \subset \quad \bigcup_{i=j}^{k} E_i$$

is a deformation retract for $j = k, k-1, \cdots, 1$. Furthermore, it also follows that $S_{k+1} \subset E_{k+1}$ is a deformation retract.

The algebraic implication of the preceding paragraph is that the homomorphism $\pi_1(E_k) \to \pi_1(E_{k+1})$ is equivalent to the homomorphism $\pi_1(S^1 \times \{0\} \times \{k+1\}) \to \pi_1(S_{k+1})$ induced by inclusion. Since the composite of this inclusion with the retraction $E_{k+1} \to S^1 \times \{0\} \times \{k+2\}$ has degree d, it follows that all the homomorphisms of fundamental groups in this paragraph are equivalent to multiplication by d_k on $\mathbb{Z}$.$\blacksquare$

Note. Some of the mappings constructed in (iii) do not preserve basepoints particularly well, but this will not cause problems because in all cases the spaces are homotopy equivalent to S^1. This means that their fundamental groups are abelian and there are unique change of basepoint isomorphisms.

(iv) Each subset S_k is compact, and since E_k is a quotient of a finite union of subsets homeomorphic to $S_1, \cdots, S_k$, it follows that E_k is also compact. Furthermore, if $K \subset E$ is compact then its image in $[0, \infty)$ will also be compact, and since this image is contained in some closed interval $[0, M]$ it follows that $K \subset E_M$ for some M. The statements about the topology of E all follow from the fact that E is homeomorphic to T (but we shall not need these in the next step, which is the last one).$\blacksquare$

(v) We have shown that the diagram of fundamental group maps is the same as the algebraic diagram considered in (i), so by (i) it is only necessary to check that it satisfies properties (2) and (3) in (i). These follow from the Compact Supports Property for fundamental groups (Proposition VIII.1.12) and the fact that every compact subset of E is contained in some E_k.$\blacksquare$

(vi) Everything will go through if we modify the definition of the integer sequence d_k; specifically, if we are only interested in fractions which are monomials in S we can take d_k to be the product of the first k primes in S if $|S| \geq k$ and taking d_k to be the product of all the primes in S if $|S| < k$. If we now define c_k as before to be $d_1 \cdots d_{k-1}$ for the new sequence $\{d_k\}$, then $S^{-1}\mathbb{Z}$ will be the union of the sets $c_k^{-1}\mathbb{Z}$.$\blacksquare$