Detailed verification that the map in moebius.pdf is 1-1

Note that $r(t,s) > 0$ everywhere, and $0 \le \theta(t,s) \le 2\pi$.
This means that (t,s) and (t',s') go to the same point in $\mathbb{R}^3 \Longleftrightarrow$

$$r(t,s) = r(t',s')$$
$$\theta(t,s) = \theta(t',s') + 2k\pi \quad k = 0 \text{ or } \pm 1$$
$$z(t,s) = z(t',s')$$

The middle condition can be rewritten as either

$$\theta(t,s) = \theta(t',s') \quad ① \quad \text{or} \quad \left\{ \begin{matrix} \theta(t,s) = 0 \\ \theta(t',s') = 2\pi \end{matrix} \right\} ② \quad \text{or vice versa.} \quad ③$$

Case 1 We have $2\pi t = 2\pi t'$, so $t = t'$.
Also $s \cos \pi t = 1 - r(s,t) = 1 - r(s',t') = s' \cos \pi t'$

$$s \sin \pi t = z(s,t) = z(s',t') = s' \sin \pi' t'$$

and since $t = t'$ this implies $s' = s$.

Cases 2 and 3 We only do Case 2; Case 3 follows by switching the roles of (t,s) and (t',s'). — The conditions imply $t = 0$ and $t' = 1$. As in Case 1 we get $s \cos \pi \cdot 0 = s' \cos \pi \cdot 1$; since $\cos \pi \cdot 0 = \cos 0 = 1$ and $\cos \pi = -1$, it follows that $s' = -s$, or equivalently $t = 0$, $t' = 1$ and $s' = -s$.

To summarize, (s,t) and (s',t') map to the same point of $\mathbb{R}^3$ $\Longleftrightarrow$ (s,t) and (s',t') belong to the same $\mathcal{R}$-equivalence class. Therefore the map

$$[0,1]\times\left[-\tfrac{1}{3},\tfrac{1}{3}\right] \longrightarrow \mathbb{R}^3$$

constructed in <u>moebius.pdf</u> passes to a map $M \longrightarrow \mathbb{R}^3$ which is continuous and <u>1</u>-<u>1</u> (it is easy to check the map is constant on each $\mathcal{R}$-equivalence class!).