Spaces which are not topological manifolds

The purpose of this document is to prove that certain topological spaces in Appendix B of `fundgpnotes2014.pdf` are not topological manifolds for any choice of n. The proofs require input from Mathematics 205B; specifically, we shall use the notion of local homology groups at a point, which is defined and studied to some extent in Section VII.1 of the file

`http://math.ucr.edu/~res/math205B-2012/algtop-notes.pdf`

and is the subject of some exercises in the related file

`http://math.ucr.edu/~res/math205B-2012/exercises05.pdf`

(see Exercises 1 and 2). In some cases it is possible to give proofs which do not involve the concepts developed in Mathematics 205B, but usually the arguments are much longer and less illuminating.

We need to start by describing the local homology groups of a topological n-manifold; this result is very similar to Theorem VII.1.7 in `algtop-notes.pdf`.

PROPOSITION. *If M is a topological n-manifold ($n \geq 1$) and $x \in M$, then the local homology group $H_k(M, M - \{x\})$ is isomorphic to $\mathbb{Z}$ if $k = n$ and is zero otherwise.*

Proof. Let U be an open neighborhood of x which is homeomorphic to an open subset in $\mathbb{R}^n$. Then the proof of the previously cited Theorem VII.1.6 in `algtop-notes.pdf` implies that the groups $H_k(U, U - \{x\})$ are given as in the statement of the proposition, and the excision isomorphisms $H_k(U, U - \{x\}) \to H_k(M, M - \{x\})$ implies that the same conclusion holds for the groups $H_k(M, M - \{x\})$.■

Note that the proposition implies the following analog of Theorem VII.1.6:

COROLLARY. (Invariance of dimension for topological manifolds.) *If M is a topological manifold of dimension m and N is a topological manifold of dimension n, then M and N are homeomorphic only if $m = n$.*

This follows because local homology groups are invariant under homeomorphism (see Proposition VII.1.4 in the previously cited notes) and the local homology groups of M and N are nonzero if and only if the dimension is m for M and n for N. By topological invariance, these numbers must be the same.■

We shall now analyze the examples from Appendix B in `fundgpnotes2014.pdf` and show that they do not satisfy the conclusion of the proposition for any value of n.

The figure eight space. This example is homeomorphic to a graph Γ with vertices A, B, C, D, E, F, G and the following edges:

$$AB\ , \quad BC\ , \quad CD\ , \quad DA\ , \quad DE\ , \quad EF\ , \quad FG\ , \quad GD$$

In particular, the vertex D lies on four edges. By Proposition VII.1.6 in `algtop-notes.pdf`, we then have $H_1(\Gamma, \Gamma - \{D\}) \cong \mathbb{Z}^3$. Since $\mathbb{Z}^3$ is never the local homology group of a topological manifold, it follows that the figure eight space Γ cannot be a topological n-manifold for any value of n.

The closed half-space $\mathbb{R}^n_+$ consisting of all points in $\mathbb{R}^n$ whose last coordinate is nonnegative. By the proposition it will suffice to show that the local homology groups $H_k(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{\mathbf{0}\})$ are zero for all k. This will follow from the long exact homology sequence of a pair if we can show that both $\mathbb{R}^n_+$ and $\mathbb{R}^n_+ - \{\mathbf{0}\}$ are contractible spaces. We know that $\mathbb{R}^n_+$ is contractible because it is a convex subset of $\mathbb{R}^n$, and we shall prove the contractibility of $\mathbb{R}^n_+ - \{\mathbf{0}\}$ by constructing an explicit homeomorphism between this space and $D^{n-1} \times (0, \infty)$. The contractibility of the latter space will then imply the contractibility of $\mathbb{R}^n_+ - \{\mathbf{0}\}$.

To construct this homeomorphism, write a point of $\mathbb{R}^n \cong \mathbb{R}^{n-1} \times \mathbb{R}$ as an ordered pair (y, u), where $y \in \mathbb{R}^{n-1}$ and $u \in \mathbb{R}$. Consider the mapping h from $D^{n-1} \times (0, \infty)$ to $\mathbb{R}^n_+ - \{\mathbf{0}\}$ which sends (v, t) to $\left(t\,v, t\sqrt{1 - |v|^2}\right)$. In geometric terms, the mapping h sends D^{n-1} to the upper hemisphere of S^{n-1} and is defined elsewhere by radial extension.

We shall show that h is a homeomorphism by constructing its inverse explicitly. In other words, if $(v, s) \neq (0, 0)$ in $\mathbb{R}^n_+$, we want to solve the equation

$$(x, s) \quad = \quad \left(t\,v, t\sqrt{1 - |v|^2}\right)$$

for t and v in terms of x and s uniquely, and we want the formulas for t and v to be continuous in terms of x and s. The appropriate solutions are given by

$$t \quad = \quad \sqrt{|x|^2 + s^2}\;, \qquad v \quad = \quad \frac{1}{\sqrt{|x|^2 + s^2}} \cdot x$$

where both expressions are meaningful and continuous because $(x, s) \neq (0, 0)$ implies that the expression $\sqrt{|x|^2 + s^2}$ is always positive. As noted previously, this implies that $\mathbb{R}^n_+ - \{\mathbf{0}\}$ is contractible and hence that the local cohomology groups $H_k(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{\mathbf{0}\})$ are zero for all k.■

FURTHER RESULTS. The methods of this document give complete information on the local homology groups $H_k(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{x\})$ for all $x \in \mathbb{R}^n_+$. Recall that the defining condition for the latter subspace is that the last coordinate be nonnegative.

PROPOSITION. *In the notation of the previous paragraph, the local homology groups are given as follows:*

(i) If the last coordinate of x is positive, then $H_k(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{x\})$ is isomorphic to $\mathbb{Z}$ if $k = n$ and is zero otherwise.

(ii) If the last coordinate of x is zero, then $H_k(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{x\})$ is zero for all k.

Note that by excision a similar result is true if $\mathbb{R}^n_+$ is replaced by an open subset $U \subset \mathbb{R}^n_+$.

Before proving the proposition, we note one consequence:

COROLLARY. *Let U and V be open subsets of $\mathbb{R}^n_+$. If h is a homeomorphism from U to V, then h sends the set of all points with positive last coordinate in U to the set of all points with positive last coordinate in V, and likewise for the sets of all points with in U and V whose last coordinates are equal to zero.*

This result plays a key role in the study of **manifolds with boundary**. Formally, a topological n-manifold with boundary is a Hausdorff space X such that every point in X has an open neighborhood which is homeomorphic to an open subset in $\mathbb{R}^n_+$.

Proof of the Corollary. By Proposition VII.1.4, if $x \in U$ then $H_k(U, U - \{x\}) \cong H_k(V, V - \{h(x)\})$ for all k. In particular, since the sets of all points with positive last coordinates are precisely those points for which the local homology groups are not all zero, it follows that if x has a positive last coordinate then $h(x)$ must also have a positive last coordinate, and if x has a last coordinate equal to zero then $h(x)$ must also have a last coordinate equal to zero.■

Proof of the Proposition. Suppose first that the last coordinate of x is positive. Then x lies in $W = \mathbb{R}^n_+ - (\mathbb{R}^{n-1} \times \{0\})$, which is an open subset of $\mathbb{R}^n$. By excision the local homology groups at x (viewed as a point in $\mathbb{R}^n$) are isomorphic to $H_k(W, W - \{x\})$, and by Theorem VII.1.7 in `algtop-notes.pdf` we know that $H_k(W, W - \{x\})$ is zero if $k \neq n$ and $\mathbb{Z}$ if $k = n$.

Now suppose that the last coordinate of x is zero, and let T be the translation of $\mathbb{R}^n$ such that $T(y) = y + x$ for all y. Then one can check directly that T maps $\mathbb{R}^n_+$ to itself, and the resulting homeomorphism T_+ from $\mathbb{R}^n_+$ to itself maps $H_k(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{\mathbf{0}\})$ isomorphically to $H_k(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{x\})$. Since the groups $H_k(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{\mathbf{0}\})$ are all zero, the same must be true for the groups $H_k(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{x\})$.■

MANIFOLDS WITH BOUNDARY. The preceding results play a key role in the definition of **manifolds with boundary**. Formally, a topological n-manifold with boundary is a Hausdorff space X such that every point in X has an open neighborhood which is homeomorphic to an open subset in $\mathbb{R}^n_+$. It follows from the preceding discussion that if X is a topological n-manifold with boundary then the local homology groups $H_k(X, X - \{x\})$ are either zero for all k — in which case we say that x is a *boundary point* — or else $H_k(X, X - \{x\})$ is zero if $k \neq n$ and $\mathbb{Z}$ if $k = n$ — in which case we say that x is an *interior point*. The sets of boundary points and interior points are denoted by ∂X and $\mathrm{Int}\, X$ respectively; these uses of the words "boundary" and "interior" differ from the usual meanings of interior and boundary in point set topology, but if both types of concepts are needed one can distinguish them by means of modifying phrases like "interior/boundary in the sense of point set topology" or "manifold interior/boundary."

We can interpret the previously defined concept of a topological n-manifold as a special case of a topological n-manifold with boundary because $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^{n-1} \times (0, \infty)$ (hence every open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^n_+$. Since the n-dimensional local homology groups at a point are always nontrivial for a topological n-manifold in the original sense, it follows that the latter can be viewed as a topological n-manifold with boundary for which the boundary is the empty set.

The following result is fundamental in the study of manifolds with boundary.

THEOREM. *Let X be a topological n-manifold with boundary. Then the manifold interior of X is a topological n-manifold in the original sense, and the manifold boundary ∂X is a topological $(n-1)$-manifold with empty boundary in the original sense. Furthermore, the manifold interior $\mathrm{Int}\, X$ is open and dense in X, and the manifold boundary ∂X is closed in X.*

Proof. Let $x \in X$, and let U be an open neighborhood of x in X which is homeomorphic to an open subset U' in $\mathbb{R}^n_+$. Let $h : U' \to U$ be an explicit homeomorphism, and choose x' so that $h(x') = x$.

Since the local homology groups of x in X are isomorphic to $H_*(U, U - \{x\})$ by excision and the latter are isomorphic to the groups $H_*(U', U' - \{x'\})$ by topological invariance, it follows that $x \in \partial X$ if $x' \in (\mathbb{R}^{n-1} \times \{0\}) \cap U'$ and $x \in \mathrm{Int}\, X$ if $x' \in \big(\mathbb{R}^{n-1} \times (0, \infty)\big) \cap U'$. Since X is the union of the disjoint subsets $\mathrm{Int}\, X$ and ∂X and the conditions on x' are mutually exclusive, the converses of the assertions in the preceding sentence are also true.

Continuing with the setting of the preceding paragraph, suppose now that $x \in \mathrm{Int}\, X$. Since h is an open mapping (it is a homeomorphism onto an open subset), it follows that the image under h of the open subset $U' - (\mathbb{R}^{n-1} \times \{0\}) \subset \mathbb{R}^n_+$ — which is $U - \partial X$ — is an open subset of X which is contained in $\mathrm{Int}\, X$. Therefore each point of $\mathrm{Int}\, X$ is contained in an open subset X which is entirely contained in $\operatorname{Int} X$, and therefore the latter must be an open subset of X. Since the boundary and interior of X are complementary subsets, this also implies that ∂X is closed in X.

Continuing with the setting of the preceding paragraph, its reasoning shows that if x lies in $\operatorname{Int} X$ then the open neighborhood $U - \partial X$ of x in X is homeomorphic to an open subset of $\mathbb{R}^{n-1} \times (0, \infty)$, and since the latter is open in $\mathbb{R}^n$ it follows that $\operatorname{Int} X$ is a topological n-manifold in the original sense.

We shall now show that ∂X is a topological $(n-1)$-manifold with empty boundary. Suppose now that $x \in \partial X$ and let U be an open neighborhood of x in X which is homeomorphic to an open subset U' in $\mathbb{R}^n_+$. Let $h : U' \to U$ be an explicit homeomorphism, and choose x' so that $h(x') = x$; we then know that $x' \in \mathbb{R}^{n-1} \times \{0\}$. We claim that the restriction of h to $U_0 = U' \cap \left(\mathbb{R}^{n-1} \times \{0\}\right)$ maps the latter homeomorphically to an open subset of ∂X; if true, this claim will show that ∂X is a topological $(n-1)$-manifold with empty boundary. By construction we know that $h|U_0$ is continuous and 1–1, and by the second paragraph of the proof we know that the image of $h|U_0$ is contained in ∂X. Therefore it will suffice to show that if W_0 is an open subset of U_0 then h maps W_0 to an open subset of ∂X.

To prove the assertion in the preceding sentence, start by writing $W_0 = W \cap \left(\mathbb{R}^{n-1} \times \{0\}\right)$ for some open subset W in $\mathbb{R}^n_+$. Since $W_0 \subset U_0 \subset U'$, it follows that we can replace W by $W' = W \cap U'$ in the preceding set-theoretic equation. By construction and the reasoning of the second paragraph in the proof, it follows that h maps W' homeomorphically onto an open subset $V \subset U$ and that W_0 is sent onto $h[V] \cap \partial X$. But this means that the image of W_0 under h is an open subset of ∂X, which is what we needed to show in order to prove that ∂X was a topological $(n-1)$-manifold in the original sense.

The only statement in the theorem which still requires proof is that $\operatorname{Int} X$ is dense in X, and this reduces to showing that every point of ∂X is a limit point of $\operatorname{Int} X$. Once again we shall use the setting described at the beginning of the proof, with $x \in \partial X$ and $h(x') = x$. If we write $x' = (y, 0) \in \mathbb{R}^{n-1} \times \{0\}$, then since U' is open in $\mathbb{R}^n_+$ and $x' \in U'$ there is some $\delta > 0$ such that $0 < t < \delta$ implies $(y, t) \in U'$, and from this we conclude that x' is a limit point of $U' \cap \left(\mathbb{R}^{n-1} \times (0, \infty)\right)$. Since h maps the latter homeomorphically to $U \cap \operatorname{Int} X$, it follows that x is a limit point of $U \cap \operatorname{Int} X$ and hence x is also a limit point of $\operatorname{Int} X$.∎

Finally, we state the following result without proof and include an interesting consequence.

TOPOLOGICAL COLLAR NEIGHBORHOOD THEOREM. *If X is a topological n-manifold with boundary, then there is an open neighborhood U of ∂X and a homeomorphism $c : \partial X \times [0, 1) \to U$ such that $c(y, 0) = y$ for all $y \in \partial X$.*

Before proceeding to the corollary, we note that $U \cap \operatorname{Int} X$ is the image of $\partial X \times (0, 1)$ with respect to c.

COROLLARY. *If X is a topological n-manifold with boundary, then the inclusion of* $\operatorname{Int} X$ *in X is a homotopy equivalence.*

Sketch of proof. Let c define a collar neighborhood as in the theorem, and let $X^* \subset X$ be the complement of $c\,[\,[0, h)\,]$, where $0 < h < 1$; we can use the collar neighborhood neighborhood theorem to show that there is a homeomorphism φ from X to X^* such that the composite of φ with the inclusion $j : X^* \subset X$ is homotopic to the identity. For example, on the collar neighborhood, shrink everything from $\partial X \times [0, 1)$ to $\partial X \times [h, 1)$ using the map from $[0, 1]$ to $[h, 1]$ which takes $[0, \frac{1}{3}]$ to $[h, \frac{1}{3}(2h + 1)]$ linearly, takes $[\frac{1}{3}, \frac{1}{3}(h + 2)]$ to $[\frac{1}{3}(2h + 1), \frac{1}{3}(h + 2)]$ linearly, and takes $[\frac{1}{3}(h + 2), 1]$ to itself by the identity.

It is elementary to verify that $X \times \{h\}$ is a strong deformation retract of $X \times (0, h]$ and $X \times [0, h]$. If we glue these deformation retract data to the identity on X^* along $c[X \times \{h\}]$, we see that X^* is a strong deformation retract of both $\mathrm{Int}\, X$ and X. Furthermore, it also follows that a homotopy inverse to the inclusion $\mathrm{Int}\, X \subset X$ is given by the composite $X \to X^* \subset \mathrm{Int}\, X$.■

References for the proof of the Collar Neighborhood Theorem. The first two papers (by Brown and Connelly) given proofs in the compact case, and the methods can be extended to cover all metrizable (hence paracompact) manifolds with boundary using the methods in the paper by Rubin (specifically, Lemma 1, which is implicit in the paper by Michael).

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