A continuous family of product metrics

Throughout the discussion below, (X, d^X) and (Y, d^Y) will denote fixed metric spaces. Furthermore, unless explicitly stated otherwise, $\mathbf{z}_\alpha = (x_\alpha, y_\alpha)$ will denote a point in the Cartesian product $X \times Y$

We shall give a detailed verification that for each real number $p \geq 1$ the function

$$d_p(\mathbf{z}_1, \mathbf{z}_2) \quad = \quad \left(d^X(x_1, x_2)^p \ + \ d^Y(y_1, y_2)^p\right)^{1/p}$$

defines a metric on $X \times Y$, and that these metrics have the following basic properties:

(1) *If $p > q \geq 1$, then $d_q \leq d_p$; this holds if p and q are real numbers and also if $p = \infty$ (where the d_∞ metric is defined as in Exercise 5.7 on page 42 of Sutherland).*

(2) *We have*

$$\lim_{p \to \infty} d_p \quad = \quad d_\infty \ .$$

The discussions in `product-metrics1.pdf` and `product-metrics2.pdf` explain why d_p is a metric space when $X = Y = \mathbb{R}$ with the standard metric, and we shall assume this result here. It will also be helpful to verify (1) and (2) in that special case before considering a product of two arbitrary metric spaces.

LEMMA. *Let u and v be real numbers such that $u \geq v \geq 0$, let $\mathbf{w} = (u, v)$, and for each real number $p \geq 1$ let*

$$|\mathbf{w}|_p \quad = \quad (u^p \ + \ v^p)^{1/p} \ .$$

Then the following hold:

(1) *If $p > q \geq 1$, then $|\mathbf{w}|_p \ \leq \ |\mathbf{w}|_q$ (hence the p-norm is a nonincreasing function of p).*

(2) *The limit of $|\mathbf{w}|_p$ as $p \to \infty$ is equal to $u = |\mathbf{w}|_\infty$ (where the latter is defined as in Sutherland and the previously cited documents).*

Proof. We begin with (1). If $v = 0$ then the definitions immediately imply that $|\mathbf{w}|_p \ = \ u \ = \ |\mathbf{w}|_q$, so equality holds in these special cases. Therefore we shall assume $v > 0$ (hence also $u > 0$) from now on. Suppose now that $u > v$ and write $(u, v) = (cs, ct) = c\,\mathbf{w}_0$, where $s^q + t^q = 1$ and $c = |\mathbf{w}|_q$. Then we must have $0 < u, v < 1$, so that $s^p + t^p < s^q + t^q = 1$ because $g(p) = y^p$ is a strictly decreasing function of p if $0 < y < 1$. It follows that

$$|\mathbf{w}|_p \quad = \quad c\,|\mathbf{w}_0|_p \ \ c\,(s^p + t^p)^{1/p} \quad < \quad c\,(s^q + t^q)^{1/p} \quad = \quad c \quad = \quad |\mathbf{w}|_q$$

because $f(x) = x^{1/p} = \exp(\log_e x/p)$ is a nondecreasing function for $x > 0$. This proves (1).■

We shall now prove (2). As in the preceding case, if $v = 0$ then we have $|\mathbf{w}|_p = u$ for $1 \leq p \leq \infty$, so the limit statement is true for trivial reasons. Assume now that $v > 0$ (hence also $u > 0$). Since $u \geq v > 0$, let $t = v/u$, so that $|\mathbf{w}|_p = u\,(1 + t^p)^{1/p}$. The conclusion is equivalent to

$$\lim_{p \to \infty} (1 + t^p)^{1/p} \quad = \quad 1 \qquad \text{if} \quad 0 \ < \ t \ \leq \ 1 \ .$$

Taking logarithms, we see that this limit statement is equivalent to

$$\lim_{p \to \infty} \frac{\log_e\,(1 + t^p)}{p} \quad = \quad 0 \qquad \text{if} \quad 0 \ < \ t \ \leq \ 1$$

and the latter is true because $1/p$ goes to 0 as $p \to \infty$ and $0 < t \leq 1$ implies that $0 < \log_e\,(1+t^p) \leq \log_e\ 2$, so that the limit formula follows from the Squeeze Principle for limits (see page 6 of `solutions01w14.pdf` for a statement of this principle).■

Proof(s) of the main result(s)

Given $\mathbf{z}_i = (x_i, y_i) \in X \times Y$ for $i = 1, 2, 3$, let $u_{i,j} = d^X(x_i, x_j)$ and $v_{i,j} = d^Y(y_i, y_j)$, and let $\alpha_{i,j} \in \mathbb{R}^2$ be given by $(u_{i,j}, v_{i,j})$. Then our definitions yield the identity $d_p(\mathbf{z}_1, \mathbf{z}_2) = |\alpha_{i,j}|_p$.

The nonnegativity and symmetry properties of d_p are immediate consequences of the corresponding results for d^X and d^Y, and if $d_p(\mathbf{z}_1, \mathbf{z}_3) = 0$ then $u_{1,3}^p + v_{1,3}^p = 0$, which happens if and only if each summand is zero, which in turn happens if and only if $\mathbf{z}_1 = \mathbf{z}_2$. Therefore it is only necessary to verify that the Triangle Inequality holds for d_p.

The Triangle Inequalities for d^X and d^Y imply that the inequalities $u_{1,3} \leq u_{1,2} + u_{2,3}$ and $v_{1,3} \leq v_{1,2} + v_{2,3}$, and since $(a^p + b^p)^{1/p}$ is an increasing function of a and b, we have the following chain of inequalities:

$$d_p(\mathbf{z}_1, \mathbf{z}_3) \;=\; \left(u_{1,3}^p + v_{1,3}^p\right)^{1/p} \;\leq\; \left((u_{1,2} + u_{2,3})^p \;+\; \left(v_{1,2}^p + v_{2,3}\right)^p\right)^{1/p} \;=\; |\alpha_{1,2} + \alpha_{1,3}|_p$$

Since $|\cdots|_p$ defines a distance on $\mathbb{R}^2$ we know that the right hand side of this expression is less than or equal to

$$|\alpha_{1,2}|_p \;+\; |\alpha_{1,3}|_p \;=\; d_p(\mathbf{z}_1, \mathbf{z}_2) \;+\; d_p(\mathbf{z}_1, \mathbf{z}_3)\ .$$

If we concatenate (string together) these inequalities, we obtain the Triangle Inequality for d_p.■

The verification of (1) and (2) for the d_p metrics is now straightforward. Since

$$d_p(\mathbf{z}_1, \mathbf{z}_2) \;=\; |\alpha_{1,2}|_p$$

and the right hand side is a nonincreasing function of p by the first part of the Lemma, the left hand side is also a nondecreasing function of p, so that $p > q$ implies $d_p \leq d_q$. Turning to the limit identity, by the Lemma we know that

$$d_p(\mathbf{z}_1, \mathbf{z}_2) \;=\; |\alpha_{1,2}|_p \;\longrightarrow\; |\alpha_{1,2}|_\infty \;=\; d_\infty(\mathbf{z}_1, \mathbf{z}_2)$$

so the limit of the d_p metrics is equal to the d_∞ metric.

COROLLARY. *The metrics d_p, for $1 \leq p \leq \infty$, define the same topology on $X \times Y$.*

Proof. This follows from Proposition 6.34 in Sutherland (see page 70) and the inequalities

$$\tfrac{1}{2} \cdot d_q \;\leq\; \tfrac{1}{2} \cdot d_1 \;\leq\; d_\infty \;\leq d_p \;\leq\; d_q \;\leq\; d_1 \;\leq\; 2 \cdot d_\infty \;\leq\; 2 \cdot d_p$$

which hold for all p, q such that $1 \leq q \leq p < \infty$.■