TOPOLOGICAL ASPECTS OF PROJECTIVE SPACES

For the most part, the notes `pg-all.pdf` concentrate on the aspects of projective geometry which are closely related to linear algebra. However, for many purposes it is necessary to view real and complex projective spaces as topological spaces. The purpose of this discussion is to define the standard topological structures on $\mathbb{RP}^n$ and $\mathbb{CP}^n$ and to establish some of their fundamental properties. In particular, we want to verify that if $\mathbb{F}$ is the real or complex numbers then the set of ordinary points in $\mathbb{FP}^n$ is topologically equivalent to $\mathbb{F}^n$ and that that projective collineations of $\mathbb{FP}^n$ are homeomorphisms with respect to the standard topologies on these objects.

We shall use the following book as a reference for undergraduate level point set topology:

W. A. Sutherland. *Introduction to Metric and Topological Spaces* (Second Edition). Oxford University Press, 2008.

There is also a companion website for this book

`http://www.oup.com/uk/booksites/content/9780199563081/`

which contains a great deal of useful supplementary material.

Digression on quotient spaces

It is often useful to think of the finite rings $\mathbb{Z}_n$ of integers mod n as quotient rings of the integers, viewing the elements of the former as the equivalence classes of integers with respect to the relation $x \equiv y \bmod n$ if and only if n evenly divides $y - x$. More generally, if we are given an equivalence relation $\mathcal{R}$ on a set X, we may view the set $X/\mathcal{R}$ of equivalence classes as a quotient set of X obtained by dividing out by the equivalence relation. There are many other examples of this sort in mathematics, where factoring out by an equivalence relation yields a type of quotient structure associated to the original object. In particular, since coordinate projective n-spaces over a field are constructed by taking equivalence classes of nonzero vectors in $\mathbb{F}^{n+1}$, we can view projective spaces as quotient objects.

From this perspective it is natural think of $\mathbb{FP}^n$ as a topological quotient space of the standard topological structure on $\mathbb{F}^{n+1} - \{\mathbf{0}\}$, with the equivalence relation given as above.

Formal Definition. If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then the standard topology on $\mathbb{FP}^n$ is the quotient topology on the latter obtained from the usual topology on $\mathbb{F}^{n+1} - \{\mathbf{0}\}$ and the quotient projection π to $\mathbb{FP}^n$ viewed as the equivalence classes of the relation $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{y}$ is a nonzero scalar multiple of $\mathbf{x}$.

Proposition 15.10 in Sutherland (see pp. 161–164) describes several alternative characterizations of the standard topology on $\mathbb{RP}^2$; this result has a straightforward generalization to $\mathbb{RP}^n$ for all positive integers n (except for part (e), whose generalization is more complicated but also less useful).

At various points in our discussion we shall need general results about quotient spaces beyond those in Sutherland, so we shall state and prove them here.

In set theory, there is a simple 1–1 correspondence between equivalence relations on a set X and onto functions from X to some other set Y. If we have an equivalence relation $\mathcal{R}$, then the projection $X \to X/\mathcal{R}$ — which sends $x \in X$ to its equivalence class — is onto by construction. On the other hand, if we have an onto function $f : X \to Y$, then we can define an equivalence relation

$\mathcal{R}(f)$ with $x \sim y$ if and only if $f(x) = f(y)$, and it is an elementary exercise to check that there is a 1–1 correspondence between $X/\mathcal{R}(f)$ and Y which sends the equivalence class $[x]$ of x to $f(x)$. By construction, if $\pi : X \to X/\mathcal{R}$ is the projection from $x \in X$ to its equivalence class $[x]$, then π is continuous if $X/\mathcal{R}$ is equipped with the quotient topology, and in the study of quotient topologies it is often necessary to consider the following converse question:

Recognition Problem for quotient topologies. *Suppose that we are given a continuous onto mapping $f : X \to Y$, and define $\mathcal{R}_f$ as above. What sorts of conditions on f are sufficient to guarantee that the $1-1$ correspondence from $X/\mathcal{R}(f)$ to Y is a homeomorphism?*

The first step in analyzing such questions is to observe that the map from $X/\mathcal{R}(f)$ to Y is always continuous.

PROPOSITION 1. *Let $g : X \to W$ be a continuous mapping of topological spaces, let $\mathcal{R}$ be an equivalence relation on X, let $\pi : X \to X/\mathcal{R}$ be the mapping sending x to its equivalence class $[x]$ with respect to $\mathcal{R}$, and assume that if $x \sim y$ with respect to $\mathcal{R}$ then $g(x) = g(y)$. Then there is a unique continuous function $\overline{g} : X/\mathcal{R} \to Y$ such that $g = \overline{g} \circ \pi$.*

Proof. This can be extracted from two results in Sutherland. The existence of a unique mapping of sets $\overline{g} : X/\mathcal{R} \to Y$ follows from Proposition 15.3 (see p. 155), and the continuity of $\overline{g}$ follows from Proposition 15.8 (see p. 157) and the continuity of g.■

COROLLARY 2. *In the setting of the Recognition Problem, the $1-1$ correspondence from $X/\mathcal{R}$ to Y is continuous.*

This follows because the 1–1 correspondence is given by the map $\overline{f}$ associated to f in the proposition.■

It is not difficult to give examples where the 1–1 correspondence $\overline{f}$ is not a homeomorphism; one trivial way of doing so is to take the equivalence relation $\mathcal{E}$ on X given by $x = y$ and to define f to be the identity map from X with the discrete topology to X with the indiscrete topology (so that f is not a homeomorphism if X has more than one element). It is a routine exercise to verify that the quotient space projection from X to $X/\mathcal{E}$ is a homeomorphism (the latter is essentially X with the discrete topology), and this implies that the map $\overline{f}$ cannot be a homeomorphism.

One can also construct examples of onto maps $f : X \to Y$ where X and Y are subsets of $\mathbb{R}^n$ but $\overline{f}$ is not a homeomorphism. Perhaps the simplest example is the map f from $[0,1)$ to the unit circle S^1 in $\mathbb{R}^2$ which sends t to $(\cos 2\pi t, \sin 2\pi t)$ with $\mathcal{R}$ given by the equality relation $\mathcal{E}$ such that $t \sim t' \quad \Leftrightarrow \quad t = t'$ (in which case π is again a homeomorphism and f again does not have a continuous inverse). However, there are many instances in which additional information about f will guarantee that $\overline{f}$ is a homeomorphism, and here is a simple but extremely useful result along these lines:

PROPOSITION 3. *Suppose that $f : X \to Y$ is continuous and onto, and suppose in addition that f is* EITHER *an open mapping (images of open subsets are open)* OR *a closed mapping (images of closed subsets are closed). Then the canonical continuous mapping $\overline{f}$ is a homeomorphism.*

As suggested by Definition 15.6 in Sutherland (see p. 157), a continuous mapping satisfying the conclusion of Proposition 3 is frequently called a *quotient map* of topological spaces.

Proof. We need to prove that $\overline{f}^{-1}$ is continuous, and since the $\overline{f}$ is 1–1 and onto we can do this by showing that either (i) $\overline{f}$ sends open subsets of $X/\mathcal{R}$ to open subsets of Y, or (ii) $\overline{f}$ sends closed subsets of $X/\mathcal{R}$ to closed subsets of Y. By construction of the quotient topology a subset of $X/\mathcal{R}$ is open or closed if and only if its inverse image in X is open or closed respectively (the openness statement is true by definition, and the statement about closed subsets is Exercise 15.7 on page 172 of Sutherland), and the arguments for the two cases will parallel each other.

Suppose first that f is open. If A is open in $X/\mathcal{R}$, then $\pi^{-1}[A] \subset X$ is open, and therefore

$$\overline{f}[A] \quad = \quad f\left[\pi^{-1}[A]\right]$$

will be an open subset of Y. On the other hand, if f is closed, the same reasoning (with "closed" replacing "open" everywhere) shows that if A is closed in $X/\mathcal{R}$ then $\overline{f}[A]$ will be a closed subset of Y.∎

Finally, if we are given a continuous mapping of topological spaces $h : X_1 \to X_2$ and equivalence relations $\mathcal{R}_i$ on X_i for $i = 1$ or 2, then we shall need a criterion for concluding that h passes to a continuous mapping of quotient spaces from $X_1/\mathcal{R}_1$ to $X_2/\mathcal{R}_2$.

PROPOSITION 4. *Suppose that $h : X_1 \to X_2$ is a continuous mapping of topological spaces, let $\mathcal{R}_2$ and $\mathcal{R}_2$ be equivalence relations on X_1 and X_2 respectively, and let $pi_i : X_i \to X_i/\mathcal{R}_i$ be the quotient projection. If for all u and v such that $u \; \mathcal{R}_1 \; v$ we also have $h(u) \; \mathcal{R}_1 \; h(v)$, then there is a unique continuous mapping h^* from $X_1/\mathcal{R}_1$ to $X_2/\mathcal{R}_2$ such that $\pi_2 \circ h = h^* \circ \pi_1$.*

Proof. Apply Proposition 1 to $g = \pi_2 \circ h$.∎

COMPLEMENT TO PROPOSITION 4. *The mapping h^* is also the unique mapping of sets such that $\pi_2 \circ h = h^* \circ \pi_1$.*

This follows from the argument proving Proposition 4 and the purely set-theoretic Proposition 15.3 in Sutherland which was cited earlier.∎

Continuity and projective collineations

One justification for the definitions of the topologies on the projective spaces $\mathbb{FP}^n$ is that projective collineations from $\mathbb{FP}^n$ to itself are always continuous. In fact, one can say substantially more.

THEOREM 5. *If $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and n is a positive integer, then every projective collineation T from $\mathbb{FP}^n$ to itself is a homeomorphism.*

Proof. We shall derive this as a consequence of the Fundamental Theorem of Projective Geometry (see Chapter VI in `pg-all.pdf`) and Proposition 4 together with its complement.

By the Fundamental Theorem of Projective Geometry the projective collineation T is defined by an invertible $(n+1) \times (n+1)$ matrix A over $\mathbb{F}$. Specifically, if $X \in \mathbb{FP}^n$ has homogeneous coordinates ξ, then $T(X)$ has homogeneous coordinates $A\xi$. In the framework of Proposition 4 and its complement, if $\pi : \mathbb{F}^{n+1} - \{\mathbf{0}\} \to \mathbb{FP}^n$ is the quotient projection and $L(A)$ is the map from $\mathbb{F}^{n+1} - \{\mathbf{0}\}$ to itself given by A, then $\pi \circ L(A) = T \circ \pi$. Therefore we can use Proposition 4 and its complement to conclude that T is continuous. By definition we know that a projective collineation is 1–1 onto, so all that remains is to verify that T^{-1} is continuous. But this follows immediately because T^{-1} is also a projective collineation.∎

Recall that the standard inclusion j of affine n-space in projective n-space is given as follows: If $\theta : \mathbb{F}^n \to \mathbb{F}^{n+1} - \{\mathbf{0}\}$ is the map which sends a vector with coordinates $(t_1, \cdots, t_n)$ to $(t_1, \cdots, t_n, 1)$ then $j = \pi \circ \theta$.

PROPOSITION 6. *If $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and n is a positive integer, then the mapping j is $1-1$, continuous and open (and hence j maps $\mathbb{F}^n$ homeomorphically onto its image, which is open).*

Proof. The results of `pg-all.pdf` imply that j is 1–1, and the factorization $j = \pi \circ \theta$ implies that j is a composite of continuous maps and hence continuous itself, so it only remains to show that j is sends open subsets to open subsets.

Suppose that $U \subset \mathbb{F}^n$ is open; we want to show that

$$\pi^{-1}\big[\, j[U]\big] \quad = \quad \pi^{-1}\big[\, \pi[U \times \{1\}]\big] \quad = \quad \{(x,t) \in \mathbb{F}^n \times (\mathbb{F} - \{0\}) \mid t^{-1}x \in U\}$$

is open in $\mathbb{F}^{n+1} - \{\mathbf{0}\}$. For simplicity of notation we shall denote the displayed subset by W, and we shall construct a homeomorphism of the open set

$$\mathbb{F}^n \times (\mathbb{F} - \{0\}) \quad \subset \quad \mathbb{F}^{n+1} - \{\mathbf{0}\}$$

to itself such that W is the image of the open subset $U \times (\mathbb{F} - \{0\})$. We can define such a homomorphism by $\varphi(x,t) = (tx, t)$; its inverse is the map sending (y,t) to $(t^{-1}y, t)$. Since φ is a homeomorphism which sends the open subset $U \times \mathbb{F} - \{0\}$, to W, it follows that the latter is open and hence the mapping j is open.■

In particular, Proposition 6 implies that the set of points at infinity in $\mathbb{FP}^n$ is a closed subset. However, if we combine this proposition with the preceding theorem, we obtain a much stronger conclusion.

PROPOSITION 7. *If $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and n is a positive integer, then* every *hyperplane $H \subset \mathbb{FP}^n$ is a closed subset. Furthermore, if H is a hyperplane then its complement is dense.*

Proof. As noted above, the hyperplane H_∞ of points at infinity, which is the complement of the image of j, is an open subset because j is an open mapping, and as such its image is open. But if H is an arbitrary hyperplane, then there is a projective collineation of $\mathbb{FP}^n$ to itself which sends H_∞ to H. Since a projective collineation is a homeomorphism and homeomorphisms map closed subsets to closed subsets, it follows that H is a closed subset of $\mathbb{FP}^n$.

To prove that the complement of H is dense, we need to show that if $X \in \mathbb{FP}^n$ then every open neighborhood of X contains a point in the complement. As in the preceding discussion, the idea will be to prove this in selected cases and then to extend the result to the general situation using projective collineations. Since the statement in the first sentence is clearly true if $X \in \mathbb{FP}^n - H$, it will be enough to verify the result when $X \in H$.

Consider the special case where $X = j(\mathbf{0})$ and H is the projective extension of the ordinary hyperplane defined by the linear equation $x_1 = 0$. Suppose that U is an open neighborhood of X in $\mathbb{FP}^n$, and let $V = U \cap \mathrm{Image}(j)$. For this example there clearly are points in $(\mathbb{FP}^n - H) \cap V = V - H$; it is only necessary to take a point of the form $j(t\,\mathbf{e}_1)$ where $|t|$ is sufficiently small.

Turning to the general situation, let X_0 and H_0 be the examples in the preceding paragraph. By the Fundamental Theorem of Projective Geometry there is a projective collineation T such that $T(X_0) = X$ and $T[H_0] = H$. Since T is a homeomorphism, it follows that given $X \in H$ and an open neighborhood U of X then the intersection $(\mathbb{FP}^n - H) \cap U$ will be nonempty because $U_0 = T^{-1}[U]$ is an open neighborhood of X_0 and by the preceding paragraph $(\mathbb{FP}^n - H_0) \cap U_0$ is known to be nonempty.■

The power of Theorem 5 combined with the Fundamental Theorem of Projective Geometry is also apparent in the next result, which concerns topological properties of the spaces $\mathbb{FP}^n$.

THEOREM 8. *If $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and n is a positive integer, then $\mathbb{FP}^n$ is compact and Hausdorff.*

Proof. We shall prove compactness first. Let $d = 1$ or 2 depending upon whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, so that S^{dn+d-1} is the unit sphere in $\mathbb{F}^{n+1}$. Since every nonzero vector in $\mathbb{F}^{n+1}$ can be represented as a product $c\,\mathbf{u}$ where $\mathbf{u}$ lies on the unit sphere and c is a positive real number, it follows that the restriction $\pi|S^{dn+d-1}$ maps onto $\mathbb{FP}^n$. Since S^{dn+d-1} is compact, it follows that its image, which is $\mathbb{FP}^n$, must also be compact.

We shall now prove that the Hausdorff condition is satisfied; as noted in Exercise 15.6 from Sutherland (pp. 171–172), a quotient space of a metric space need not be Hausdorff, so this is a nontrivial issue.

Given two points X and Y in $\mathbb{FP}^n$ we need to construct disjoint open neighborhoods for them. If both points lie in the image of j, then this is easy because $\mathbb{F}^n$ is Hausdorff, and by Proposition 6 its image must also be Hausdorff. Consequently, if $\mathbf{x}$ and $\mathbf{y}$ satisfy $j(\mathbf{x}) = X$ and $j(\mathbf{y}) = Y$, then one can find disjoint neighborhoods U' and V' of $\mathbf{x}$ and $\mathbf{y}$, and their images $U = j[U']$ and $V = j[V']$ will be disjoint open neighborhoods of X and Y in $\mathbb{FP}^n$.

Suppose now that X and Y are arbitrary points. By the conclusions of the preceding paragraph it will be enough to find a projective collineation which maps X and Y to points in the image of j; *i.e.*, neither X nor Y lies on the hyperplane at infinity. Since there are projective collineations taking this hyperplane to all the other hyperplanes, it will be enough to show that *given two distinct points in $\mathbb{FP}^n$, there is a hyperplane H which contains neither of them.*

We can prove the assertion in the preceding sentence as follows. If ξ and η are homogeneous coordinates for X and Y respectively, then ξ and η are linearly independent because $X \neq Y$. Choose vectors $\alpha_1, \cdots, \alpha_{n-1}$ so that $\mathcal{B} = \{\alpha_1, \cdots, \alpha_{n-1}, \xi, \eta\}$ is a basis for $\mathbb{F}^{n+1}$. Then the vectors $\alpha_1, \cdots, \alpha_{n-1}$ and $\xi + \eta$ are linearly independent (verifying this is an elementary exercise based upon the linear independence of $\mathcal{B}$, and therefore there is a unique hyperplane H containing the the associated points $\pi(\alpha_1), \cdots, \pi(\alpha_{n-1})$ and $\pi(\xi+\eta)$. We claim that neither X nor Y lies in H. If, say, $X \in H$ then the set of vectors $\mathcal{B}' = \{\alpha_1, \cdots, \alpha_{n-1}, \xi + \eta, \xi\}$ is linearly dependent; however, the linear independence of $\mathcal{B}$ implies the linear independence of $\mathcal{B}'$ (this is another elementary exercise), and therefore we must have $X \notin H$. Similar considerations imply that $Y \notin H$. As indicated earlier, this suffices to complete the proof that $\mathbb{FP}^n$ is Hausdorff.■

The next result implies that projective spaces also satisfy the second Default Hypotheses in Section VIII.9 of `fundgps-notes.pdf`:

THEOREM 8A. *If $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and n is a positive integer, then $\mathbb{FP}^n$ is a topological dn-manifold, where $d = 1$ if $\mathbb{F} = \mathbb{R}$ and $d = 2$ if $\mathbb{F} = \mathbb{C}$. In particular, $\mathbb{FP}^n$ is locally arcwise connected.*

Proof. It suffices to show that every point $X \in \mathbb{FP}^n$ has an open neighborhood which is homeomorphic to $\mathbb{F}^n$. If X is an ordinary point which lies in the image of the canonical inclusion j, then this follows from Proposition 6. If $X \in \mathbb{FP}^n$ is not in the image of j and $\xi = (x_0, \cdots, x_{n+1})$ is a set of homogeneous coordinates for X, then $x_{n+1} = 0$ but there is some $k < n+1$ such that $x_k \neq 0$. Let T_k be the projective collineation of $\mathbb{FP}^n$ arising from the linear transformation on $\mathbb{F}^{n+1}$

which switches the k^{th} and last coordinates (and leaves the remaining coordinates untouched). Then X will lie in the image V_k of $T_k \circ j$; since T_k is a homeomorphism and j is an open mapping it follows that V_k is a open neighborhood of X which is homeomorphic to $\mathbb{F}^n$, and by the first sentence of the paragraph this suffices to complete the proof.■

COROLLARY 9. (Compare Sutherland, Proposition 15.10.(a) and (b), page 161.) *If $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and n is a positive integer, then $\mathbb{FP}^n$ is homeomorphic to the quotient of the unit sphere in $\mathbb{F}^{n+1}$, which is S^{dn+d-1} modulo the equivalence relation $\mathbf{x} \sim \mathbf{y}$ if and only if there is some $z \in \mathbb{F}$ such that $|z| = 1$ and $\mathbf{x} = z\,\mathbf{y}$.*

Recall that if $\mathbb{F} = \mathbb{R}$ then the set of all z satisfying $|z| = 1$ is $\{1, -1\}$, and if $\mathbb{F} = \mathbb{C}$ then this set is all complex numbers of the form $\cos\theta + i\sin\theta$ for some θ.

Proof of Corollary 9. In the proof of the theorem we noted that $q = \pi|S^{nd+d-1}$ is onto, and one can check directly that if $\mathbf{x}, \mathbf{y} \in S^{nd+d-1}$, then $\pi(\mathbf{x}) = \pi(\mathbf{y})$ if and only if there is some $z \in \mathbb{F}$ such that $|z| = 1$ and $\mathbf{x} = z\,\mathbf{y}$ (this uses the length identity $|c\,\mathbf{v}| = |c| \cdot |\mathbf{v}|$). Therefore, if A is the quotient of the unit sphere described in the theorem, there is a continuous 1–1 onto mapping from A to $\mathbb{FP}^n$. Since A is (the continuous image of) a compact space and $\mathbb{FP}^n$ is Hausdorff, it follows from Proposition 13.26 in Sutherland (pp. 135–136) that this map is a homeomorphism.■

Remark. Using more advanced material from point set topology or differential geometry, one can prove that $\mathbb{FP}^n$ is metrizable. — Here is a proof using point set topology (the necessary background is not given here, but it can be found in nearly every graduate level point set topology textbook): By the Urysohn Metrization Theorem, a compact Hausdorff space is metrizable if and only if it is second countable. By construction $\mathbb{FP}^n$ is a union of finitely many metrizable sets; namely, the complements of the hyperplanes defined by the homogeneous linear equations $x_i = 0$, where $1 \leq i \leq n+1$. Since it is an elementary exercise to prove that if a space X is a union of finitely many second countable open subsets, then X itself is second countable, this means that $\mathbb{FP}^n$ is second countable (and hence metrizable).

We have seen that a hyperplane in $\mathbb{FP}^n$ is closed, and by continuity we know that the zero set of a nontrivial polynomial in n variables is a closed subset of $\mathbb{F}^n$. We shall use the preceding results to prove a similar result for the zero sets of homogeneous polynomials in $\mathbb{FP}^n$. Recall that if $f(x_1, \cdots, x_{n+1})$ is a **homogeneous** polynomial (a sum of monomials with the same degree $d > 0$, so that $f(c\,\mathbf{x}) = c^d f(\mathbf{x})$ for all c and $\mathbf{x}$) and its zero set is $V(p) \subset \mathbb{F}^{n+1}$, then $W = V(p) - \{\mathbf{0}\}$ satisfies $W = \pi^{-1}\big[\pi[W]\big]$, and hence we can think of $V^{\text{proj}}(p) = \pi[W]$ as the projective zero set for the polynomial p. Of course, if p has degree 2 then this set is a hyperquadric.

PROPOSITION 10. *In the setting described above, $V^{\text{proj}}(p)$ is a closed subset of $\mathbb{FP}^n$.*

Proof. By Exercise 15.7 in Sutherland, it suffices to prove that the inverse image of $V^{\text{proj}}(p) = \pi[W]$ is closed in $\mathbb{F}^{n+1} - \{\mathbf{0}\}$, and this follows because

$$\pi^{-1}\left[V^{\text{proj}}(p)\right] \quad = \quad \pi^{-1}\big[\pi[W]\big] \quad = \quad W \quad = \quad V(p) - \{\mathbf{0}\}$$

and $V(p)$ is a closed subset of $\mathbb{F}^{n+1}$.■

We also have an analog of the Sparseness Theorem in `quadrics1.pdf`:

PROPOSITION 11. *In the setting described above, if p is a nonzero homogeneous polynomial of positive degree, then the interior of $V^{\text{proj}}(p)$ in $\mathbb{FP}^n$ is empty, and $\mathbb{FP}^n - V^{\text{proj}}(p)$ is dense in $F\mathbb{P}^n$.*

Proof. (This argument will use the Sparseness Theorem cited above.) The density statement follows from the emptiness of the interior exactly as in the proof of the Sparseness Theorem, so it will suffice to prove that the interior of $V^{\mathrm{proj}}(p)$ in $\mathbb{FP}^n$ is empty.

Suppose that U is a nonempty open subset of $\mathbb{FP}^n$ which is contained in $V^{\mathrm{proj}}(p)$. Since π is onto and continuous, it follows that $\pi^{-1}[U]$ is a nonempty open subset of $\mathbb{F}^{n+1}$ which is contained in $V(p) = \pi^{-1}[V^{\mathrm{proj}}(p)] \cup \{\mathbf{0}\}$. Therefore the Sparseness Theorem from `quadrics1.pdf` implies that p is the zero polynomial.■

Of course, both of the preceding results apply to hyperquadrics in $\mathbb{FP}^n$.

FINAL REMARKS. One can define a similar topology on $\mathbb{FP}^n$ if $\mathbb{F}$ is the quaternions, and everything through Corollary 9 can be generalized; however, since multiplication in the quaternions is not commutative it is necessary to be work with right vector spaces over a division ring rather than vector spaces over a field (see the first appendix in `pg-all.pdf` for a discussion of linear algebra over a division ring). There is also a minor complication involving the Fundamental Theorem of Projective Geometry: If $\mathbb{F}$ is a field, then this result states the existence of a unique projective collineation with certain properties, but if $\mathbb{F}$ is only a division ring (*i.e.*, the multiplication is not commutative) then one still has an existence statement because the latter only involves concepts from linear algebra. Fortunately, we only need existence and not uniqueness in the arguments given above.