# Proper Maps and Universally Closed Maps

Reinhard Schultz

Continuous functions defined on compact spaces generally have special properties. For example, if $X$ is a compact space and $Y$ is a Hausdorff space, then $f$ is a closed mapping. There is a useful class of continuous mappings called *proper*, *perfect*, or *compact* maps that satisfy many properties of continuous maps with compact domains. In this note we shall study a few basic properties of these maps. Further information about proper maps can be found in Bourbaki [$\mathrm{B}_2$], Section I.10), Dugundji [D] (Sections XI.5–6, pages 235–240), and Kasriel [K] (Sections 95 and 105, pages 214–217 and 243–247). There is a slight difference between the notions of perfect and compact mappings in [$\mathrm{B}_2$], [D], and [K] and the notion of proper map considered here. Specifically, the definitions in [D] and [K] include a requirement that the maps in question be surjective. The definition in [$\mathrm{B}_2$] is entirely different and corresponds to the notion of *universally closed* map discussed later in these notes; the equivalence of this definition with ours for a reasonable class of spaces is proved both in [$\mathrm{B}_2$] and in these notes (see the section *Universally closed maps*).

## PROPER MAPS

If $f : X \to Y$ is a continuous map of topological spaces, then $f$ is said to be *proper* if for each compact subset $K \subset Y$ the inverse image $f^{-1}[K]$ is also compact.

**EXAMPLE.** *Suppose that $A$ is a finite subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n - A \to \mathbb{R}^m$ is a continuous function such that for all $a \in A$ we have*

$$\lim_{x \to a} |f(x)| = \lim_{x \to \infty} |f(x)| = \infty.$$

*Then $f$ is proper.*

**Proof.** If $K \subset \mathbb{R}^m$ is compact, then $K$ is contained in some large disk $D$. The limit condition at $\infty$ implies that $f^{-1}[D]$ is contained in some large disk $D' \subset R^n$; furthermore the limit conditions at all the points $a_i \in A$ imply that $f^{-1}[D]$ is also contained in the complement of $E = D' - \cup N_i$, where $N_i$ is a small open disk neighborhood with center $a_i$. But $E$ is compact and $f^{-1}[K]$ is a closed subset of $\mathbb{R}^n$ that is contained in $E$; it follows that $f^{-1}[K]$ is compact.■

**IMPORTANT SPECIAL CASE.** *Suppose $n = m = 2$ and $f(z) = p(z)/q(z)$, where $p$ and $q$ are complex polynomials with deg $p >$ deg $q$. Then $f$ satisfies the limit condition and therefore is proper.*

The first result, which is fairly standard, shows that proper maps of certain not necessarily compact spaces have some important properties in common with continuous maps of compact Hausdorff spaces.

**THEOREM 1.** *Suppose that $X$ and $Y$ are Hausdorff spaces with $Y$ either locally compact or metrizable, and let $f : X \to Y$ be continuous. Then $f$ is proper if and only if $f$ is closed and for each $y \in Y$ the set $f^{-1}[\,\{y\}\,]$ is compact.*

**Proof:** Suppose that $f$ is closed and inverse images of one point sets are compact; we claim that inverse images of arbitrary compact subsets are compact.

Let $K$ be a compact subset of $Y$, and let $\mathcal{F} = \{F_\alpha\}$ be a collection of closed subsets of $f^{-1}[K]$ with the finite intersection property. Without loss of generality, we may assume that $\mathcal{F}$ is closed under finite intersections, for if $\mathcal{F}\,'$ is the family of all finite intersections of sets in $\mathcal{F}$ then $\mathcal{F}\,'$ is closed under finite intersections and the intersection of all the sets in $\mathcal{F}$ equals the intersection of all the sets in $\mathcal{F}\,'$. Since $f$ is closed, the subsets $f[F_\alpha]$ are closed in $Y$; since $F_\alpha \subset f^{-1}[K]$ is true by assumption, it follows that $f[F_\alpha] \subset K$. Combining these observations, we see that $f[F_\alpha]$ is a compact subset of $K$. But the family of closed sets $f(F_\alpha)$ also has the finite intersection property because

$$\varnothing \neq f\big[\,F_{\alpha_1}\,\big] \;\cap\; \cdots \;\cap\; f\big[F_{\alpha_k}\,\big]) \subset f\big[\,F_{\alpha_1}\,\big] \;\cap\; \cdots \;\cap\; f\big[\,F_{\alpha_k}\,\big]\;.$$

Therefore $\varnothing \neq\; \cap\, f\big[\,F_\alpha\,\big]$ by compactness of $K$. Let $y$ be a point in the intersection. Since the family $\mathcal{F}$ is closed under finite intersections, for all $\alpha_1,\;\cdots\;,\alpha_n$ we have

$$f^{-1}\big[\,\{y\}\;\cap\;F_{\alpha_1}\;\cap\;\cdots\;\cap\;F_{\alpha_n}\,\big] \neq \varnothing\;.$$

Therefore the family $\{\,f^{-1}(\{y\})\;\cap\;F_\alpha\}$ has the finite intersection property. But $f^{-1}\big[\,\{y\}\,\big]$ is compact, and therefore

$$\varnothing \;\neq\; f^{-1}\big[\,\{y\}\,\big]\;\cap\;(\cap_\alpha\,F_\alpha)\;\subset\;\cap_\alpha\,F_\alpha\;.$$

It follows that the set $f^{-1}[K]$ is compact. $\square$

Suppose now that $f$ is proper and $Y$ is locally compact or metrizable. If $F \subset X$ is a closed subset, then it is immediate that $f|F$ is proper. Therefore it suffices to prove that if $f$ is proper then $f[X]$ is closed in $Y$ (because $f[F] = f|F\,[F]\,$).

Assume first that $Y$ is locally compact. Let $y$ be a point in the closure of $f[X]$, and let $K$ be a compact neighborhood of $y$. Then $f^{-1}[K]$ is a nonempty compact set, and $f|\,f^{-1}[K]$ a closed mapping. Therefore $f\big[\,f^{-1}[K]\big] = f[X]\;\cap\;K$ is a closed set. But by construction $y$ is a limit point of this set, and consequently $y \in f[X]\;\cap K\;\;\subset f[X]$. $\square$

Assume now that $Y$ is a metric space. Let $F$ be closed in $X$, and let $\{x_n\}$ be a sequence of points in $F$ such that $\lim f(x_n) = y$. Let $C_n$ be the compact set $\{y, f(x_n), f(x_{n+1}), ...)$, so that $f^{-1}[C_n]$ is a nonempty nested sequence of compact sets. Therefore by compactness we have that

$$\varnothing \neq\; \cap_n\;(\,f^{-1}[C_n]\;\cap F\,)\;=\;(\,f^{-1}\big[\cap_n\,C_n\,\big]\,)\;\cap\;F\;=\;f^{-1}\big[\,\{y\}\,\big]\;\cap\;F\;.$$

Therefore $y \in f[F]$ must hold, and consequently $f[F]$ must be closed in $Y$.$\blacksquare$

**Remark,** The $(\Longleftarrow)$ implication does not require an extra condition on $Y$. The $(\Longrightarrow)$ implication is valid more generally if $Y$ is a $k$-space (see [D], Section XI.9, especially XI.9.3 on p. 248). Here is the proof that $f$ is closed under this hypothesis: The set $f[X]$ is closed if and only if

$f[X] \cap K$ is compact for all compact subsets $K \subset Y$. But $f[X] \cap K = f\left[f^{-1}[K]\right]$, and this set is compact because $f$ proper $\Longrightarrow f^{-1}[K]$ is compact and $f^{-1}[K]$ compact $\Longrightarrow f\left[f^{-1}[K]\right]$ compact. $\square$

**THEOREM 2.** *Suppose that $f : X \to Y$ is a proper map and $B \subset Y$. Then $f|f^{-1}[B]$ is proper. Conversely, if $\{B_\alpha\}$ is either an open covering or a finite closed covering of $Y$ and each of the maps $f|f^{-1}[B_\alpha]$ is proper, then $f$ is also proper.*

**Proof.** ( $\Longrightarrow$ ) Let $f_B = f|f^{-1}[B]$. If $K$ is a compact subset of $B$ then $f_B^{-1}[K] = f^{-1}[K]$; but the latter is compact since $f$ is proper, and therefore it follows that $f_B$ is proper.

( $\Longleftarrow$ ) Let $f_\alpha = f|f^{-1}[B_\alpha]$, and let $K \subset Y$ be compact. Suppose that $\{B_\alpha\}$ is an open covering of $Y$. Then by compactness $K$ is contained in a finite union $B_{\alpha_1}, \cup \cdots \cup B_{\alpha_k}$. Let $V_{\alpha_j} = K \cap B_{\alpha_j}$; then the sets $\{V_{\alpha_j}\}$ define a finite open covering of $K$, and by Munkres, Theorem 36.1, pp. 225–226 (especially Step 1), we can find closed (hence compact) subsets $F_{\alpha_j} \subset V_{\alpha_j}$ such that $K = \cup_j F_{\alpha_j}$. Therefore it follows that $f^{-1}[K] = \cup_j f_{\alpha_j}^{-1}\left[F_{\alpha_j}\right]$. But each map $f_{\alpha_j}^{-1}$ is proper, and hence it follows that each subset of the form $f_{\alpha_j}^{-1}\left[F_{\alpha_j}\right]$ is compact. Since a finite union of compact sets is compact, it follows that $f^{-1}[K]$ is compact.

Now suppose that $\{B_\alpha\}$ is a finite closed covering. Let $K_\alpha = K \cap B_\alpha$; then each set $K_\alpha$ is compact because each set $B_\alpha$ is closed in $Y$. Since each map $f_\alpha$ is proper we know that each set $f_\alpha^{-1}\left[K_\alpha\right] = f^{-1}(K_\alpha)$ is compact. But $f^{-1}[K] = \cup_\alpha f_\alpha^{-1}\left[K_\alpha\right]$, and therefore $f^{-1}[K]$ is compact because a finite union of compact sets is compact.■

**Remark.** The theorem also holds for *locally finite* closed coverings.

**THEOREM 3.** *Let $f_\alpha : X_\alpha \to Y_\alpha$ be continuous for each $\alpha \in J$, and let*

$$\prod f_\alpha : \prod X_\alpha \longrightarrow \prod Y_\alpha$$

*be the associated map of product spaces. Then $\prod f_\alpha$ is proper if and only if each $f_\alpha$ is proper.*

**Proof.** ( $\Longrightarrow$ ) Let $\beta \in J$ be arbitrary, and for each $\alpha \neq \beta$ choose $x_\alpha \in X_\alpha$. Furthermore, for each $\alpha \neq \beta$ let $y_\alpha = f_\alpha(x_\alpha)$. Let $i_\beta : X_\beta \to X_\beta \times \prod_{\alpha\neq\beta}\{x_\alpha\}$ be the identity on the $\beta$ factor and constant on the other factors, and let $j_\beta : Y_\beta \to Y_\beta \times \prod_{\alpha\neq\beta}\{y_\alpha\}$ be the identity on the $\beta$ factor and constant on the other factors; then $i_\beta$ and $j_\beta$ are closed subspace inclusions and

$$\left(\prod f_\alpha\right) \circ i_\beta \;=\; j_\beta \circ f_\beta \;.$$

Let $K \subset Y_\beta$ be compact. Then the formula above implies that

$$i_\beta\left[f_\beta^{-1}[K]\right] \;=\; \left(\prod f_\alpha\right)^{-1}[\,j_\beta[K]\,]$$

and since the product map is proper it follows that $i_\beta\left[f_\beta^{-1}[K]\right]$ is compact. But $i_\beta$ is a homeomorphism onto a closed subset, and therefore a subset $B \subset X_\beta$ is compact if and only if $i_\beta[B]$ is compact. Therefore it follows that $f_\beta^{-1}[K]$ is compact. $\square$

( $\Longleftarrow$ ) Suppose that $K$ is compact in $\prod Y_\alpha$. Then $L_\alpha = \pi_\alpha[L]$ is a compact subset of $Y_\alpha$. By Tychonoff's Theorem the product $\prod L_\alpha$ is compact. It is immediate that $\left(\prod f_\alpha\right)^{-1}[K]$ is a closed subset of $\prod f_\alpha^{-1}[L_\alpha]$. But the factors $f_\alpha^{-1}[L_\alpha]$ are all compact because the maps $f_\alpha$ are proper, and therefore the product is compact by Tychonoff's Theorem. The observations of the preceding two sentences combine to show that $\left(\prod f_\alpha\right)^{-1}[K]$ is compact.■

**THEOREM 4.** *Let $f : X \to Y$ and $g : Y \to Z$ be continuous.*
($i$) *If $f$ and $g$ are proper, then $g \circ f$ is also proper.*
($ii$) *If $g \circ f$ is proper, then $f$ is proper.*
($iii$) *If $g \circ f$ is proper and $f$ is onto, then $g$ is proper.*

**Proof.** ($i$) Let $K$ be a compact subset of $Z$. Then $g^{-1}[K]$ is a compact subset of $Y$ because $g$ is proper; but $f$ is also proper, and therefore $(g \circ f)^{-1}[K] = f^{-1}\left[g^{-1}[K]\right]$ is also compact. □

($ii$) Let $C$ be a compact subset of $Y$. Then $g[C]$ is compact, and therefore $f^{-1}\left[g^{-1}\left[g[C]\right]\right])$ is a compact subset of $X$. Since $C \subset g^{-1}[g[C]]$ and $f^{-1}[C]$ is a closed subset of $X$, it follows that $f^{-1}[C]$ is a closed subset of the compact set $f^{-1}\left[g^{-1}\left[g[C]\right]\right]$, and therefore $f^{-1}[C]$ is compact. □

($iii$) Let $K$ be a compact subset of $Z$. Since $g \circ f$ is proper the set $(g \circ f)^{-1}[K] = f^{-1}\left[g^{-1}[Z]\right]$ is compact. However, $f$ is onto and therefore $B = f\left[f^{-1}[B]\right]$ for all $B \subset Y$. It follows that $g^{-1}[K] = f\left[f^{-1}\left[g^{-1}[K]\right]\right]$, and the set on the right hand side is compact because it is the image of the compact set $(g \circ f)^{-1}[K] = f^{-1}\left[g^{-1}[K]\right]$ under the continuous function $f$.■

## UNIVERSALLY CLOSED MAPS

Proper maps arise naturally in many analytic and geometric contexts (compare [$B_1$] and [Se]). For these purposes it is convenient to have an alternate characterization of proper maps; in fact, the definition in [$B_2$] takes this characterization as the definition of proper map. Since it is difficult to find an account of this outside of [$B_2$], the details are given below.

**Definition.** Let $\mathcal{A}$ be a family of Hausdorff spaces such that the following hold:
($i$) If $A \in \mathcal{A}$ and $B$ is homeomorphic to $A$ then $B \in \mathcal{A}$.
($ii$) $\mathcal{A}$ contains all one point spaces.

If $X, Y \in \mathcal{A}$, then a continuous map $f : X \to Y$ is said to be *universally closed* (with respect to $\mathcal{A}$) if for each $Z \in \mathcal{A}$ the map $f \times 1_Z : X \times Z \to Y \times Z$ is a closed mapping. Notice that if $\mathcal{B} \subset \mathcal{A}$ and $f$ is $\mathcal{A}$-universally closed, then $f$ is also $\mathcal{B}$-universally closed.

A space $X \in \mathcal{A}$ is said to be $\mathcal{A}$-*complete* if the constant map $X \to \{\text{pt.}\}$ is proper.

**LEMMA 5.** *The composite of two $\mathcal{A}$-universally closed maps is $\mathcal{A}$-universally closed.*

The proof of this is elementary.■

**THEOREM 6.** *Let $\mathcal{A}$ be one of the following families:*
($i$) *All $\mathbf{T}_{3\frac{1}{2}}$ spaces.*
($ii$) *All locally compact Hausdorff spaces.*

(*iii*) *All closed subsets of* $\mathbb{R}^n$, *where* $n$ *ranges over all nonnegative integers.*

*Then a space* $X \in \mathcal{A}$ *is* $\mathcal{A}$*-complete if and only if* $X$ *is compact.*

**Proof.** To prove the ( $\Longleftarrow$ ) implication, it suffices to show that if $X$ is compact Hausdorff and $Y$ is $\mathbf{T}_{3\frac{1}{2}}$ then the second coordinate projection $\pi_Y : X \times Y \to Y$ is a closed mapping.

Let $F$ be a closed subset of $X \times Y$, and let $W = X \times Y - F$; we claim that $W$ is open. If $y \notin \pi_Y[F]$, then for every $x \in X$ there is an open subset $U_x \times V_x \subset X \times Y$ such that $x \in U_x$, $y \in V_x$, and $U_x \times V_x \ \cap \ F \ = \ \varnothing$. The family $\{U_x\}$ forms an open covering of $X \times \{y\}$, and by compactness of $X$ there is a finite subcovering $U_{x_1}, \cdots U_{x_k}$. Let $V = \ \cap V_{x_i}$. Then $\cup\, (U_{x_i} \times V) \ \cap \ F = \varnothing$, and consequently $V \ \cap \ \pi_Y[F] = \varnothing$, so that $y \in V$. $\square$

The proof of the ( $\Longrightarrow$ ) implication separates into three cases depending on $\mathcal{A}$.

*Case 1.* Suppose that $\mathcal{A}$ is all $\mathbf{T}_{3.5}$ spaces. If $X$ is $\mathbf{T}_{3.5}$ but not compact, then $X$ is homeomorphic to a proper subset of its Stone-Čech compactification. Let $F \subset X \times \beta X$ be the graph of the embedding of $X$ in $\beta X$. Then $F$ is closed in $X \times \beta X$, but the projection $\pi_{\beta X}[X] \subset \beta X$ is a dense proper subset. $\square$

*Case 2.* Suppose that $\mathcal{A}$ is all locally compact Hausdorff spaces. The proof in Case 1 goes through if one replaces the Stone-Čech compactification $\beta X$ with the one point compactification $X^\bullet$. $\square$

*Case 3.* Suppose that $\mathcal{A}$ is all closed subsets of $\mathbb{R}^n$, where $n$ ranges over all nonnegative integers. If $X$ is a noncompact closed subset of $\mathbb{R}^k$ then its one point compactification $X^\bullet$ is a closed subspace of $(\mathbb{R}^k)^\bullet \approx S^n \subset \mathbb{R}^{k+1}$. It follows that $X^\bullet \in \mathcal{A}$, and this implies that the argument in Case 1 can be modified to apply here, with $X^\bullet$ repacing $\beta X$.$\blacksquare$

**THEOREM 7.** *Let* $\mathcal{A}$ *be either the set of locally compact Hausdorff spaces or the set of closed subspaces of the spaces* $\mathbb{R}^k$. *Then a continuous map* $f : X \to Y$ *of spaces in* $\mathcal{A}$ *is* $\mathcal{A}$*-universally closed if and only if it is proper.*

**Remark.** If $\mathcal{A}$ is the class of all Hausdorff spaces, then there are proper maps $f : X \to Y$ such that $X, Y \in \mathcal{A}$ but $f$ is not universally closed. One example is indicated in [$\mathrm{B}_2$], Exc. I.10.4: Given an uncountable family of topological spaces $X_\alpha$, define a modified product topology $\prod'$ with subbase given by all products of open sets $U_\alpha \subset X_\alpha$ where $U_\alpha = X_\alpha$ for all but at most countably many $\alpha$. If we take each $X_\alpha$ to be a two point space with the discrete topology, then it is immediate that $Y = \prod' X_\alpha$ is $\mathbf{T_3}$ and that the "identity" map $j$ from $X = \prod X_\alpha$ to $Y$ is continuous. Since $j$ is not a homeomorphism but $j$ is continuous, 1–1, and onto, it follows that $j$ cannot be closed. On the other hand, it is known that every compact subset of $Y$ is *finite* (compare [$\mathrm{B}_2$], Exc. I.9.4), and therefore $j$ is proper (in the sense of the definition in these notes).

**Proof.** ( $\Longrightarrow$ ) If $f$ is $\mathcal{A}$-universally closed and $K \subset Y$ is compact, then $K \to \{pt.\}$ is universally closed (Theorem 6), and the mapping $f_K : f^{-1}[K] \to K$ given by $f|f^{-1}[K]$ is also universally closed (a routine verification). Since the composite of universally closed maps is universally closed (see Lemma 5), it follows that the constant map $f^{-1}[K] \to \{\text{pt.}\}$ is also universally closed. But by Theorem 6 this implies that $f^{-1}[K]$ is compact. $\square$

( $\Longleftarrow$ ) If $f$ is proper, then by Theorem 3 we know that $f \times 1_Z$ is proper for all spaces $Z$ (because an identity map is always proper). Therefore by Theorem 1 the map $f \times 1_Z$ is closed.■

When working with $\mathcal{A}$-universally closed maps it is often useful to have a smaller set of test spaces $\mathcal{A}_0$ such that a map is universally closed if and only if its product with the identity map of a space in $\mathcal{A}_0$ is closed. Here is an abstract version and its most important special case(s).

**LEMMA 8.** *Let $\mathcal{A}$ be a family of Hausdorff spaces satisfying the conditions at the beginning of this section, and let $\mathcal{A}_0$ be a subfamily of $\mathcal{A}$ such that every space in $\mathcal{A}$ is homeomorphic to a closed subset of a space in $\mathcal{A}_0$. Let $X$ and $Y$ be spaces in $\mathcal{A}$. Then a continuous map $f : X \to Y$ is $\mathcal{A}$-universally closed if and only if $f \times 1_E$ is closed for all spaces $E \in \mathcal{A}_0$.*

**Important note:** It is not assumed that $X$ or $Y$ lies in the subfamily $\mathcal{A}_0$.

**Proof.** The ( $\Longleftarrow$ ) implication is immediate. To prove the ($\Longrightarrow$) implication, let $Z \in \mathcal{A}$, and let $Z$ be homeomorphic to a closed subspace of $E_0$. Without loss of generality we can assume that $Z$ is in fact a subspace of $E_0$. Then $X \times Z$ is a closed subset of $X \times E_0$, and therefore it follows that $f \times 1_{E_0}|X \times Z$ is a closed mapping. But this map factors as $j \circ (f \times 1_Z)$, where $j$ is the closed map defined by the inclusion $X \times Z \subset X \times E_0$. It follows that $f \times 1_Z$ must be a closed map.■

**Examples.** Suppose that $\mathcal{A}$ is the family of all closed subsets of $\mathbb{R}^n$, where $n$ ranges over all positive integers. Then two choices for $\mathcal{A}_0$ are the family $\mathcal{E}_{\mathbb{R}}$ consisting of all the spaces $\mathbb{R}^n$ and the family $\mathcal{E}_{\mathbb{C}}$ consisting of all the spaces $\mathbb{C}^n$.

**THEOREM 9.** *Let $\mathcal{A}$ be the family of all closed subsets of $\mathbb{R}^n$, where $n$ ranges over all positive integers, let $X, Y \in \mathcal{A}$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then a continuous map $f : X \to Y$ is proper if and only if $f \times 1_{\mathbb{F}^n}$ is closed for all positive integers $n$.*

**Proof.** By Theorem 6 the map $f$ is proper if and only if it is $\mathcal{A}$-universally closed, and by Lemma 8 this is true if and only if $f \times 1_E$ is closed for all $E \in \mathcal{E}_{\mathbb{R}}$ or $\mathcal{E}_{\mathbb{C}}$.■

In algebraic and complex analytic geometry this result is used extensively (compare [$B_1$]).

## EXERCISES

1. Suppose that $f : X \to Y$ is a continuous map of noncompact locally compact $T_2$ spaces. Let $f^\bullet : X^\bullet \to Y^\bullet$ be the map of one point comapactifications defined by $f^\bullet|X = f$ and $f^\bullet(\infty_X) = (\infty_Y)$. Prove that $f$ is proper if and only if $f^\bullet$ is continuous.

2. Prove analogs of Theorems 6 and 7 for $\mathcal{A} =$ the class of all separable metric spaces. [*Hint:* Why is every separable metric space homeomorphic to a subspace of a compact one?]

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