

The Real Number System

Preliminaries

Given the fundamental importance of the real numbers in mathematics, it is important for mathematicians to have a logically sound description of the real number system. In particular, the adoption of set theory as the basic language for mathematics means that the real numbers need to be described formally in terms of set theory. As indicated in Munkres, two ways of approaching this are as follows:

- I. One can assume the existence of the real numbers as a mathematical system with certain so-called “undefined concepts” and axioms.
- II. One can construct the real numbers explicitly from other objects in set theory; *e.g.*, the natural numbers.

From a strictly theoretical viewpoint the second alternative has an important advantage; it dramatically simplifies the total number of assumptions that are needed to set up a logical framework for mathematics. However, it has a major disadvantage for our purposes because it requires a considerable amount of effort to carry out and justify the necessary constructions, and the work itself does not shed a great deal of light on the basic topics of this course. Therefore, as in Munkres we shall choose the first alternative. A reader who is interested in the details of the “minimalist” approach to the real number system can find the details in several references (to be listed here).

The Axioms for the Real Numbers

Addition and multiplication are examples of *binary operations*. We need to describe such objects explicitly before going any further.

Definition: Given a set S , a *binary operation* on S is a function $\&$ from $S \times S$ to S . If x and y belong to S and $\&$ is a binary operation we generally write $x \& y$ for $\&(x, y)$.

The so-called “undefined entities” for the real number system are

- *(real) numbers,*
- *addition of real numbers,*
- *multiplication of real numbers,*
- *linear ordering of real numbers.*

Formally, these are given by a nonempty set \mathbf{R} (whose elements are the real numbers), a binary operation $+$ representing addition on \mathbf{R} , a binary operation \times or \cdot representing multiplication on \mathbf{R} , and a partial ordering $<$ on \mathbf{R} . Following standard mathematical practice we shall often denote $a \times b$ or $a \cdot b$ simply by ab

GROUP I: ALGEBRAIC EQUALITY AXIOMS

A-1:

(Associative Laws) For all $a, b, c \in \mathbf{R}$ we have $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

A-2:

(Commutative Laws) For all $a, b \in \mathbf{R}$ we have $a + b = b + a$ and $ab = ba$.

A-3:

(Additive and Multiplicative Identities) There exist unique and distinct numbers 0 and 1 in \mathbf{R} such that $a + 0 = a$ and $a \cdot 1 = a$ for all $a \in \mathbf{R}$.

A-4:

(Additive and Multiplicative Inverses) For each $a \in \mathbf{R}$ there exists a unique $b \in \mathbf{R}$ such that $a + b = 0$, and for each $a \in \mathbf{R}$ with $a \neq 0$ there exists a unique $c \in \mathbf{R}$ such that $c \neq 0$ and $a \cdot c = 1$.

A-5:

(Distributive Law) For all $a, b, c \in \mathbf{R}$ we have $a \cdot (b + c) = ab + ac$.

A-6:

(Multiplication by Zero) For each $a \in \mathbf{R}$ we have $a \cdot 0 = 0$.

GROUP II: INEQUALITY AXIOMS

B-1:

(Equals added to unequals) If $a, b \in \mathbf{R}$ satisfy $a > b$ then for all $c \in \mathbf{R}$ we have $a + c > b + c$.

B-2:

(Unquals multiplied by positive equals) If $a, b \in \mathbf{R}$ satisfy $a > b$ and $c > 0$ then we have $a \cdot c > b \cdot c$.

GROUP III: COMPLETENESS AXIOM

C-1:

Let S be a nonempty subset of \mathbf{R} that has an upper bound; *i.e.*, there is a real number a such that $x \leq a$ for all $x \in S$. Then S has a least upper bound; *i.e.*, there is a real number b such that $b \leq a$ for every upper bound a of the set S .

Property **A-6** is not stated explicitly in Munkres, but it is a logical consequence of the other four Algebraic Equality Axioms; we have included it here because it is so basic and would have to be taken as an axiom if it were not a formal consequence of the other properties.

Some standard concepts (negative numbers, subtraction, reciprocals, quotients, positive numbers, *etc.*) are discussed on page 31 of Munkres, and a few consequences are also listed on pages 31 and 32. Numerous other elementary consequences like the “paradoxical” identity

$$(-1) \cdot (-1) = +1$$

can be found in most undergraduate textbooks for abstract algebra. We shall mention a few additional consequences here and explain the relation between the real numbers and the positive integers from our perspective. In a subsequent section we shall show that there is essentially only one system (up to a reasonable notion of equivalence known as isomorphism) that satisfies the axioms for the real numbers.

Squares of nonzero numbers are positive: *If r is a nonzero real number, then r^2 is positive.*

Finding n -th roots of positive real numbers: *If r is a positive real number and n is a positive integer, then there is a unique positive real number x such that $x^n = r$.*

The preceding two observations imply that *a nonzero real number is positive if and only if it is the square of another (nonzero) real number.* This algebraic property plays a key role in showing that the axioms for the real numbers are algebraically rigid; *i.e.*, the only one-to-one correspondence h of the real numbers with itself such that

$$h(a + b) = h(a) + h(b), \quad h(a \cdot b) = h(a) \cdot h(b)$$

(i.e., an **automorphism**) is the identity. We shall prove this in the section on the uniqueness of the real numbers up to isomorphism. In contrast, the **conjugation map for complex numbers** sending $a + bi$ to $a - bi$ (where $a, b \in \mathbf{R}$ and $i^2 = -1$) is an automorphism of complex numbers that is **not** equal to the identity.

Real numbers and infinite decimal expressions: *Every positive real number is the sum of an infinite series of the form*

$$a_N 10^N + a_{N-1} 10^{N-1} + \dots + a_0 + b_1 10^{-1} + b_2 10^{-2} + \dots + b_k 10^{-k} + \dots$$

where each a_i and b_j is an integer between 0 and 9, and every positive real number has a **unique** expression of this form such that b_j is positive for which infinitely many values of j . Conversely, every infinite series of the form is convergent.

Of course, real numbers are generally viewed from a practical standpoint as quantities expressible by such “infinite decimal” expansions, so this result essentially justifies the usual sorts of manipulations that one performs in order to compute with real numbers. However, although such representations of real numbers are absolutely necessary for computational purposes, they are not particularly convenient for theoretical or conceptual purposes. For example, describing the reciprocal of a positive real number (even a positive integer!) explicitly by means of infinite decimal expansions is at best awkward and at worst unrealistic. Another difficulty is that these expansions are not necessarily unique; for example, the relation

$$1.0 = 0.999999\dots$$

reflects the classical geometric series formula; the representations become unique if one insists that infinitely many terms to the right of the decimal point must be nonzero, but this generates further conceptual problems. A third issue is whether one gets the same number system if one switches from base 10 arithmetic to some other base. It is natural to expect that the answer to this question is yes, but any attempt to establish this directly runs into all sorts of difficulties almost immediately. This is not purely a theoretical problem; the use of digital computers to carry out numerical computations implicitly assumes that one can work with real numbers equally well using infinite base 2 (or base 8 or 16) analogs of decimal (base 10) expansions. This discussion strongly suggests that **the formal mathematical description of real numbers should be expressed in a manner that does not depend upon a computational base.** The axiomatic description in Munkres and these notes provides a characterization of this sort.

Density of the rational numbers: *If $a, b \in \mathbf{R}$ satisfy $a > b$ then there is a rational number r such that $a > r > b$.*

Rational numbers and integers are defined on page 32 of Munkres. In particular, if $b = 0$ then one can choose r to be the reciprocal of a positive integer (write the positive rational number r as a quotient of two positive integers m/n ; if s is equal to $1/n$ then we clearly have the

inequalities $a > r \geq s > 0$).

Integral Archimedean property: If $a \in \mathbf{R}$ then there is a nonnegative integer k such that $k > a$.

Specifically, if $a < 0$ then one can take $k = 0$, and if $a > 0$ then $1/a > 0$ and by the discussion in the preceding paragraph we can find a positive integer k such that $1/a > 1/k$. Taking reciprocals, one obtains the desired inequality $k > a$.

Intervals in the Real Number System

We shall use standard notation from calculus and real variables courses to denote various open, closed and half open intervals:

The **closed interval** $[a, b]$ consists of all real numbers x such that $a \leq x \leq b$.

The **open interval** (a, b) consists of all real numbers x such that $a < x < b$.

The **half-open interval** $(a, b]$ consists of all real numbers x such that $a < x \leq b$.

The **half-open interval** $[a, b)$ consists of all real numbers x such that $a \leq x < b$.

In the second and third cases we shall replace a by $-\infty$ to denote the sets of real numbers satisfying $x < b$ and $x \leq b$ respectively, and in the second and fourth cases we shall replace b by $+\infty$ to denote the sets of real numbers satisfying $a < x$ and $a \leq x$ respectively. If we use both conventions in the second case, this expresses \mathbf{R} as the “interval” $(-\infty, +\infty)$.

Reminder. Although one can manipulate $-\infty$ and $+\infty$ in many contexts as if they were real numbers, these objects do **not** belong to the real number system and there are also contexts in which one cannot manipulate them as if they were real numbers (for example, the product of 0 and $+\infty$ cannot be defined, and likewise for the sum of $-\infty$ and $+\infty$).

The Real Numbers and the Natural Numbers

We had previously postulated the existence of the natural numbers \mathbf{N} in terms of the Peano Axioms. In Section 4 of Munkres the existence of the natural numbers is formulated as a consequence of the postulate that there is a system satisfying the axioms for the real numbers. We shall take a slightly different approach, showing that \mathbf{N} can be viewed as a subset of \mathbf{R} in a natural way. In particular, there is a unique map $e : \mathbf{N} \rightarrow \mathbf{R}$ such that

$$e(0) = 0 \quad \text{and} \quad e(\sigma(n)) = e(n) + 1$$

for all $n \in \mathbf{N}$. Specifically, this map can be defined recursively by the given formulas.

Arithmetic operations and the successor function. The following elementary identities will be needed in the uniqueness proof for the real numbers; they hold for all $m, n \in \mathbf{N}$:

$$e(m) + e(\sigma(n)) = e(m) + e(n) + 1$$

$$e(m) \cdot e(\sigma(n)) = e(m) \cdot e(n) + e(m)$$

One can also use minor variants of these identities to define addition and multiplication on \mathbf{N} abstractly in the “minimalist” approach to the real number system.