Scissors and Paste Theorem for Open Subsets

We shall prove something slightly more general than the assertion on lines -17 to -16 of page 88 in gentop-notes.pdf

Theorem. Let X be a topological space, and let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X. Let $\coprod_\alpha U_\alpha$ be the disjoint union of the sets U_α (see p. 89 of gentop-notes.pdf) and let Y be $\coprod U_\alpha$ mod the equivalence relation

$$\underset{\in U_\alpha \times \{\alpha\}}{(p,\alpha)} = \underset{\in U_\beta \times \{\beta\}}{(q,\beta)} \quad \text{if } p = q \text{ (hence } p = q \in U_\alpha \cap U_\beta\text{)}.$$

$\varphi : \coprod U_\alpha \longrightarrow X$ is defined by $\varphi(p,\alpha) = p$, then φ passes to a map $\varphi_0 : Y \to X$ which is a homeomorphism.

<u>PROOF:</u> First of all, φ_0 is continuous because φ is continuous.

Next, φ_0 is 1-1 onto. The second is true since $x \in X \Rightarrow x \in$ some $U_\alpha \Rightarrow$ $x = \varphi(x, \alpha)$. To see the second, suppose that $\varphi(p, \alpha) = \varphi(q, \beta)$. Then $\varphi(p,\alpha) = p = q$ $= \varphi(q, \beta)$, so the images of (p, α) and (q, β) in Y are the same, and hence $[p, \alpha] = [q, \beta]$ as equivalence classes in Y, so that φ_0 is 1-1.

Finally, φ_0 is open. First note that the quotient $\coprod U_\alpha \xrightarrow{H} Y$ is open; ~~for~~ it is enough to show $H \mid U_\gamma \times \{\gamma\}$ is open for all γ. But W open in $U_\gamma \times \{\gamma\} \Rightarrow H^{-1}[H[U_\gamma \times \{\gamma\}]] \cong$ $\coprod_\alpha (U_\gamma \cap U_\alpha) \times \{\alpha\}$ which is open in $\coprod U_\alpha$.

~~Finally, S open in $Y \Rightarrow H^{-1}[S] = \coprod_\alpha$ …~~

Finally, suppose that Ω is open in Y. Since H is onto we have $\varphi_0[\Omega] = \varphi[H^{-1}[\Omega]]$, and since $H^{-1}[\Omega]$ is open it suffices to prove that φ is open. The latter is true $\Longleftrightarrow$ each restriction of φ to a set $U_{\alpha_0} \times \{\alpha_0\} \subseteq \coprod U_\alpha$ is open. Since these restrictions send (x, α_0) to x, and U_{α_0} is open in X, it follows that the restriction of φ to $U_{\alpha_0} \times \{\alpha_0\}$ is open and hence φ is also open. ∎