The Seifert – van Kampen Theorem – II

The drawing below is meant to illustrate the second part of the proof of the Seifert – van Kampen Theorem, which involves constructing a homomorphism from $\pi_1(X)$ to the pushout of $\pi_1(U)$ and $\pi_1(V)$. The idea is similar to the idea in the first part of the proof: We start with a closed curve, then we decompose it into arcs which lie in either U or V, and we use this decomposition and other data to define an element of the pushout. The main issue is to verify that this group element does not depend upon any of the choices, and the final step requires us to show that if the original curve γ is homotopically trivial, then one obtains the identity element of the pushout group. We then take a basepoint preserving homotopy from γ to the constant curve, and we decompose the unit square into smaller squares (only 36 in the drawing) such that each small square maps into either U or V. In the drawing below, the original curve is given by restricting the homotopy to the bottom edge of the square, and on the other three edges the homotopy is constant. An inductive argument then reduces the proof to showing that the adjacent curves in the drawing

blue + green + blue **blue + pink + blue**

(moving from left to right) determine the same element in the pushout group.

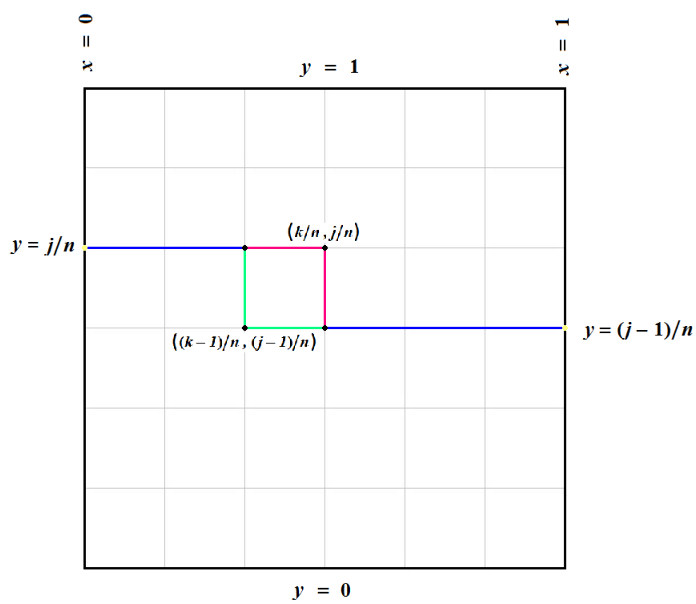

Figure 2

The verification of the inductive step follows from the fact that the pink and green curves are endpoint preserving homotopic, and the latter is true because the pink and green curves combine to form the boundary of a small square.