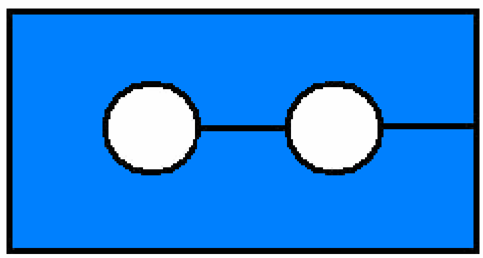
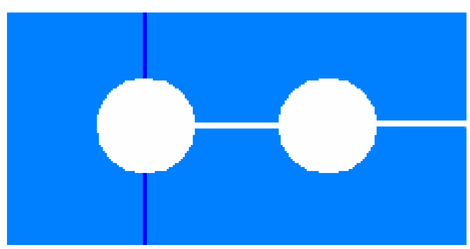
An "interesting" compactification of the plane

In the notes there was an assertion that the compactifications of the Cartesian coordinate plane $\mathbf{R}^2 \cong (0, 1)^2$ include solid rectangles with an arbitrary finite number of open holes removed. The following is a sketch of one way to prove this fact in the case of two holes. Filling in the details and extending the ideas to the case of three or more holes are left to the reader as exercises.

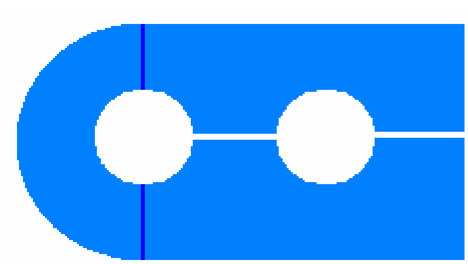
The basic idea is simple: Given a solid rectangle with two open holes removed, we need to find an open subset that is homeomorphic to $\mathbf{R}^2 \cong (0, 1)^2$. Here is an illustration of the original compact set in which the closed subset to be deleted is indicated by the thick black lines:



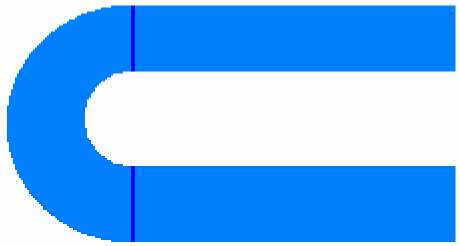
If we remove this closed set we obtain the open set illustrated below. Most of this set is shaded in light blue, with two vertical open segments shaded in dark blue. These segments will play an important role in the argument showing that the open set is homeomorphic to $\mathbf{R}^2 \cong (0, 1)^2$.



The first step in constructing the desired homeomorphism is to shrink the portion to the left of the highlighted vertical segments into an arch shaped region (in more elevated terms, half of an annulus or ring). This yields a homeomorphism of the original open set with the region illustrated below; in this picture the key vertical segments are again highlighted.



The next step is to shrink the two pieces to the right of the vertical segments into rectangles. Specifically, one does this by shrinking things vertically to the right of the segments (and not doing anything to the left), and the shrinking construction yields a homeomorphism from the figure displayed above to the horseshoe shaped region illustrated below:



Finally, one can straighten out the piece to the left of the segments into an open solid rectangle; in mathematical terms, this merely reflects the fact that a closed semicircular arc is homeomorphic to a closed interval in the real line. This transforms the horseshoe shaped region into the solid rectangle illustrated below.

This bar shaped region is clearly homeomorphic to $\mathbf{R}^2 \cong (0, 1)^2$, and therefore we have shown that the original open set is also homeomorphic to $\mathbf{R}^2 \cong (0, 1)^2$. It follows that the original compact subset represents a compactification of $\mathbf{R}^2$.

Remark

One significant point about the compact spaces formed by deleting finitely many open holes from a solid rectangle is that these spaces obtained by deleting different numbers of holes are not homeomorphic. Proving this is not possible with the methods currently at our disposal, but one can do so using the material in Munkres on fundamental groups (see Chapters **9** and **10** in particular).