# Examples of iterated subspace inclusions

The Schröder-Bernstein Theorem in set theory states that if there are injections from set $A$ to set $B$ and vice versa, then there is a bijection from $A$ to $B$, and the Axiom of Choice yields a similar conclusion if "injection" is replaced by "surjection" (since in these cases one also has injections in the opposite directions). One of the exercises in Munkres involves finding counterexamples to a topological analog if one replaces injections by topological embeddings (injections that are homeomorphisms onto their images), and here is another counterexample:

**CLAIM.** *The compact metric spaces* $[0,1] \times [0,1]$ *and* $[0,1]$ *are not homeomorphic, but each is homeomorphic to a quotient space of the other.*

**Proof.** The spaces are not homeomorphic because the complements of one point subsets are always connected in the first case but not necessarily so in the second. Furthermore, since the projection maps onto factors from $[0,1] \times [0,1]$ to $[0,1]$ are closed and surjective, it follows that the second space is homeomorphic to a quotient space of the first. Thus it only remains to show that $[0,1] \times [0,1]$ is homeomorphic to a quotient space of $[0,1]$, which will be true if there is a surjective continuous map from $[0,1]$ to $[0,1] \times [0,1]$. The standard example of such a map is described in Theorem 44.1 on pages 272–274 of Munkres; this example of a *space filling curve* was constructed by G. Peano in 1890.

**Further examples.** In fact, one can construct counterexamples to the statement about injections using compact connected metric spaces, and we shall now construct examples of this type.

Let $A$ be the solid unit disk in $\mathbb{R}^2$, and let $B$ be the union of $A$ with the line segment joining the origin to $(0,2)$. Then $A$ and $B$ are not homeomorphic for the same sorts of reasons as before (complements of one point subsets of $A$ are always connected but not necessarily so in $B$). We have the obvious embedding from $A$ to $B$ by inclusion, and the map sending $b \in B$ to $\frac{1}{2}b$ defines a topological embedding in the opposite direction. Therefore $A$ and $B$ are spaces for which there exist topological embeddings in each direction, but $A$ and $B$ are not homeomorphic.∎