Uniform continuity and completeness

The main objective is to prove the following result:

THEOREM. *Let $(X, \mathbf{d}_X)$ and $(Y, \mathbf{d}_Y)$ be metric spaces, let $f : X \to Y$ be a homeomorphism which is uniformly continuous, and suppose that Y is complete. Then X is complete.*

Note that we are not making any uniform continuity assumptions regarding the inverse map f^{-1}. However, if we do, then we have the following immediate consequence:

COROLLARY. *Let $(X, \mathbf{d}_X)$ and $(Y, \mathbf{d}_Y)$ be metric spaces, and let $f : X \to Y$ be a homeomorphism such that both f and f^{-1} are uniformly continuous. Then X is complete if and only if Y is complete.*■

If we combine this with the material in `dpmetrics.pdf`, we obtain the following additional consequence:

COROLLARY. *Let $(X_i, \mathbf{d}_i)$ be complete metric spaces for $1 \leq i \leq n$, let $p > 1$, and let $\mathbf{d}^{\langle p \rangle}$ be the associated metric on $\prod_i X_i$. Then the latter is complete with respect to the metric $\mathbf{d}^{\langle p \rangle}$.*■

Proof. Let $\{x_n\}$ be a Cauchy sequence in X; we claim that $\{\, f(x_n)\,\}$ is a Cauchy sequence in Y. Suppose that $\varepsilon > 0$ is arbitrary, and choose $\delta > 0$ such that $\mathbf{d}_X(u, v) < \delta$ implies $\mathbf{d}_Y\big(\, f(u),\ f(v)\,\big) < \varepsilon$. Then there is some N such that $m, n \geq N$ implies $d(x_m, x_n) < \delta$, and it follows that under the same conditions we have $\mathbf{d}_Y\big(\, f(x_m),\ f(x_n)\,\big) < \varepsilon$. Therefore $\{\, f(x_n)\,\}$ is a Cauchy sequence in Y.

By completeness, the sequence in the preceding sentence has a limit which we shall call y. We may now use the continuity of f^{-1} to conclude that

$$f^{-1}(y) \quad = \quad \lim_{n\to\infty} f^{-1}\big(\, f(x_n)\,\big) \quad = \quad \lim_{n\to\infty} x_n$$

which shows that the original Cauchy sequence has $f^{-1}(y)$ as a limit. Therefore X is complete.■