Describing the inverse function to $f(x) = x + e^x$

In Additional Exercise 0 for Section IV.3, at one step it is necessary to use an inverse to the function $x + e^x$ in order to describe the inverse of the example $F(x, y) = (e^x + y,\ x - y)$. Although one cannot solve directly for the inverse function $g(y)$ such that $y = x + f(x)$, this inverse can be described in terms of another much studied function known as *Lambert's W-function*. The latter is defined by the functional equation

$$w \exp(w) \quad = \quad z$$

and introductions to the basic properties of this function appear in

`http://en2.wikipedia.org/wiki/Lambert's_W_function`

and the paper by Corless *et al.* cited as a reference in that link. Although this function was first defined in the eighteenth century, there has been a great deal of renewed interest in it over the past two decades for several reasons: Advances in computer technology have made the function easier to analyze, the function is very useful in connection with computer software for symbolic manipulation of mathematical expressions, and there are several applications of this functions to other branches of science. Some additional links are listed below. The first of these includes a further link to numerous elementary exercises involving the Lambert W-function.

`http://www.apmaths.uwo.ca/~rcorless/frames/PAPERS/LambertW/`

`http://mathworld.wolfram.com/LambertW-Function.html`

`http://www.cecm.sfu.ca/publications/organic/rutgers/node34.htmlAC`

One can use Lambert's W-function to find a formula for the inverse to $x + e^x$ as follows: If we exponentiate the equation $x + e^x = y$ we obtain the equation

$$\exp(x) \cdot \exp\big(\exp(x)\big) \quad = \quad \exp(y)$$

and if we make the changes of variables $w = e^x$ and $z = e^y$ we obtain the identity $w\, e^w = z$ that defines the Lambert W-function. Solving this equation for x, we obtain the formula

$$g(y) \quad = \quad x \quad = \quad \log\big(W\,(e^y\,)\big)$$

which expresses x as a function of y very effectively, especially around the origin where one can write out a Taylor series expansion for $W(x)$ in a very neat explicit form:

$$W(x) \quad = \quad \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}\, x^n$$