Cut points and homeomorphism types

Much of this discussion is based upon excerpts from the following online reference:

http://www.math.cornell.edu/~hatcher/Top/TopNotes.pdf

One of the central problems in topology is to develop criteria for determining whether or not two topological spaces are homeomorphic. In the notes we have seen that one can sometimes use basic properties like compactness or connectedness to show that two spaces must be topologically distinct; it is only necessary to show that one space has the property in question but the other does not. We have also mentioned that more refined versions of such basic properties can also be extremely useful; for example, the half open interval $(0, 1]$ is not homeomorphic to the open interval $(0, 1)$ because the subspace $(0, 1] - \{1\}$ is connected but the complement of every point in $(0, 1)$ is disconnected.

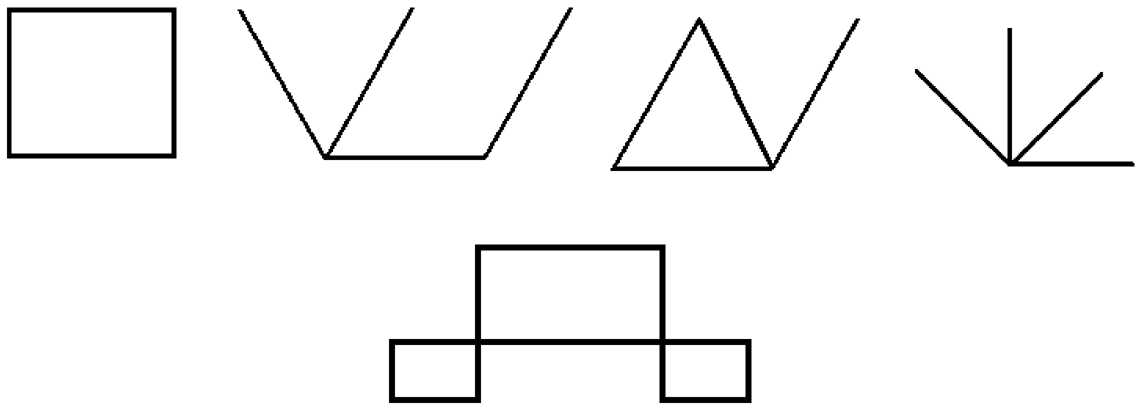
More formally, if **X** is a connected space and $x \in$ **X**, we shall say that x is a ***cut point*** of **X** if the complement **X** $- \{x\}$ is disconnected. It follows immediately that if f is a homeomorphism from one connected space **X** to another connected space **Y**, then x is a cut point of **X** if and only if $f(x)$ is a cut point of **Y**, for the restriction of f defines a homeomorphism from **X** $- \{x\}$ to **Y** $- \{f(x)\}$. In many cases one can count the numbers of cut points and non – cut points in two spaces, and if these numbers do not coincide we can conclude that the two spaces in question are not homeomorphic. Several examples appear on page 21 of the online reference cited above. Here are a few more examples based upon letters, numerals and similar characters:

character	A	B	E	H	X	8
number of cut points	infinitely many	0	infinitely many	infinitely many	infinitely many	1
number of non – cut points	infinitely many	infinitely many	3	4	4	infinitely many

There are several obvious elaborations of this idea; in particular, for every finite subset **A** of a topological space **X** one can consider the connectedness of the complement **X – A** or more generally the number of connected components in the latter. For example, at each point of the Figure **H** space the complement has at most three components, but there is a point of the Figure **X** space for which the complement has four components.

We shall conclude with an exercise adapted from the online reference cited above.

Using cut points and non – cut points, explain why no two of the following five subspaces of the plane are homeomorphic (each is a closed subspace given by the union of a finite number of closed line segments).



(Go to the next page for the solution to this problem.)

SOLUTIONS

We shall count the number of cut points and non – cut points in each case; the conclusion will follow because no two examples yield the same ordered pair of values.

In the first example there are infinitely many non – cut points and no cut points.

In the second example there are three non – cut points and infinitely many cut points.

In the third example there are infinitely many non – cut points and infinitely many cut points.

In the fourth example there are four non – cut points and infinitely many cut points.

In the fifth example there are infinitely many non – cut points and three cut points.

The drawings below illustrate the different types of points; in each case, the cut points are colored red.