Embeddings into product spaces

The main result is a generalization of Theorem 34.2 on pages 217–218 of Munkres, and the argument is similar to the proof of "Step 2" on pages 215–216 of Munkres.

Definition. Let Y be a topological space, and let $f_\alpha : Y \to X_\alpha$ be an indexed family of continuous mappings with indexing set A. If $\pi_\beta : \prod_\alpha Y_\alpha \to Y_\beta$ is projection onto the β-factor, then the basic properties of products imply there is a unique continuous mapping $F : Y \to \prod_\alpha X_\alpha$ such that $\pi_\alpha \circ F = f_\alpha$ for all α. The main result gives sufficient conditions for F to map Y homeomorphically onto its image.

THEOREM. *In the setting above, the mapping F maps Y homeomorphically onto $F[Y]$ provided the following hold:*

(i) *The family of functions f_α* separates points*: If u and v are distinct points of Y then there exists some f_α such that $f_\alpha(u) \neq f_\alpha(v)$.*

(ii) *The family of functions f_α* separates points and closed subsets*: Given $x \in Y$ and a closed subset $A \subset Y$ such that $x \notin A$, there is some f_α such that $f_\alpha(x) \notin \overline{f_\alpha[A]}$.*

Note that if X is $\mathbf{T_1}$ then the second condition implies the first.

Proof. First of all, condition (i) is equivalent to the condition that the function F is 1–1, for the condition means that if $u \neq v$ then $f_\alpha(u) = \pi_\alpha \circ F(u)$ and $f_\alpha(v) = \pi_\alpha \circ F(v)$ are unequal for some indexing variable α.

Therefore, if the first condition holds, then the map F defines a 1–1 continuous mapping onto $F[Y]$. We need to show that if the second condition also holds then F sends open subsets of Y to open subsets of $F[Y]$. It will suffice to show that for each open subset $V \subset Y$ and $x \in V$ there is an open neighborhood W of $F(x)$ in $\prod_\alpha X_\alpha$ such that $W \cap F[Y] \subset F[V]$.

We know that $x \notin X - V$ and $X - V$ is closed, so there is some α such that $f_\alpha(x) \notin \overline{f_\alpha[X - V]}$; we shall denote the closed subset in this expression by E.

Let $W = \pi_\alpha^{-1}[Y_\alpha - E]$; since E is closed in Y_α it follows that W is open. Clearly $U = W \cap F[Y]$ is open in $F[Y]$; we shall show that $z = F(x) \in U$ and $U \subset F[V]$. Since $x \in V$ is arbitrary, the usual argument shows that $F[V]$ is a union of open subsets in $F[Y]$ and hence is open in $F[Y]$, which is what we wanted to prove.

To show that $z \in U = W \cap F[Y]$, first note that $z = F(x)$ implies $z \in F[Y]$, and $F(z) \in W$ because $\pi_\alpha \circ F(z) = f_\alpha(z) \notin E$, which means that $F(z) \in W = \pi_\alpha^{-1}[Y_\alpha - E]$.

Suppose now that $y \in U = W \cap F[Y]$; it follows that $y = F(y')$ for some $y' \in Y$. Now

$$\pi_\alpha(y) \quad = \quad \pi_\alpha \circ F(y') \quad = \quad f_\alpha(y')$$

and since $y \in W$ implies that $\pi_\alpha(y) \notin Y_\alpha - E$, it follows that $f_\alpha(y') \notin Y_\alpha - E$. Since f_α maps $X - V$ into E and $f_\alpha(y') \notin E$, it follows that y' must lie in V. The latter in turn implies that $y = F(y')$ must lie in $F[V]$. This completes the proof that $U = W \cap F[Y]$ is contained in $F[V]$, and as noted above it also completes the proof of the theorem.■