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Mathematics 205A, Fall 2003, Examination 1

Point values are indicated in brackets.

1. (*i*) [*10 points*] Give an example of a continuous 1–1 onto mapping that is not a homeomorphism.

(*ii*) [*20 points*] Given a topological space X and $Y \subset X$, let $\overline{Y}$ denote the closure of Y in X. Show that the closure operation satisfies $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ for all A, $B \subset X$, and given an example for which the containment is proper.

SOLUTIONS.

For the fist part, one can take the identity map on a set S where the domain has the discrete topology and the codomain has any smaller topology (for example, the indiscrete topology, assuming $|X| \geq 2$). A second example would be the map from $[0, 1)$ to the circle S^1 taking t to $\exp 2\pi i t$.■

For the second part, note that $A, B \subset A \cap B$ implies that $\overline{a}$, $\overline{B} \subset \overline{A \cap B}$ and hence $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. To show that containmnet may be proper, let $X = \mathbf{R}$ and take A and B to be the disjoint invervals $(0, 1)$ and $(1, 2)$. The closures of these sets are the closed intervals $[0, 1]$ and $[1, 2]$, and therefore the intersection of their closures is $\{1\}$, which is not empty.■

2. [*25 points*] Given a metric space X and a point $x \in X$, show that the one-point subset $\{x\}$ is closed in X.

FIRST SOLUTION.

We shall show that the complement $X - \{x\}$ is open in X. Let $y \in X - \{x\}$, so that $y \neq x$ and $r = \mathbf{d}(y, x) > 0$. Then $N_r(y) \subset X - \{x\}$ because $r = \mathbf{d}(y, x) > 0$. Therefore the set $X - \{x\}$ must be open in X.■

SECOND SOLUTION.

Suppose that $\{a_n\}$ is a sequence in $\{x\}$ and $a = \lim_{n \to \infty} a_n$. We need to show that $a \in \{x\}$. By definition, a point p lies in $\{x\}$ if and only if $p = x$, the infinite sequence reduces to the constant sequence $a_n = x$ for all n. The limit of a constant sequence is just the constant value, so this implies that a must be equal to x and therefore $\{x\}$ must be closed.■

3. [*30 points*] Let X and Y be topological spaces satisfying the Hausdorff Separation Property. Prove that $X \times Y$ also satisfies the Hausdorff Separation Property.

SOLUTION.

Let (x_1, y_1) and (x_2, y_2) be distinct points in $X \times Y$. There are two cases depending upon whether the first or second coordinates of (x_1, y_1) and (x_2, y_2) are unequal. Since the two points are distinct at least one of these applies.

Supppose that $x_1 \neq x_2$. Then there are disjoint open subsets U_1 and U_2 in X such that $x_i \in U_i$, and $U_1 \times Y$ and $U_2 \times Y$ are disjoint open subsets of $X \times Y$ containing (x_1, y_1) and (x_2, y_2) respectively.

Supppose now that $y_1 \neq y_2$. Then there are disjoint open subsets V_1 and V_2 in Y such that $y_i \in V_i$, and $X \times V_1$ and $X \times V_2$ are disjoint open subsets of $X \times Y$ containing (x_1, y_1) and (x_2, y_2) respectively.■

4. [*15 points*] Let A and B be homeomorphic subsets of $\mathbf{R}^n$ such that A is closed and bounded. Prove that B is also closed and bounded. Give an example of two homeomorphic subsets $A,\ B \subset \mathbf{R}^n$ such that A is closed but not bounded and B is bounded but not closed.

SOLUTION.

If A is closed and bounde3d then A is compact. Since there is a homeomorphism $h : A \to B$ it follows that $B = h(A)$ is also compact, which in turn implies that B is closed and bounded.■

The easiest examples for the second part would be a bounded open interval and the entire real line. For example, the function $f(x) = \tan x$ defines a homeomorphism from $(-\pi/2,\ \pi/2)$ to $\mathbf{R}$ (its inverse is the arc tangent function).■