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Mathematics 205A, Fall 2003, Examination 2

Point values are indicated in brackets.

1. [*20 points*] Suppose we know that the system of equations $u = f(x,y)$, $v = g(y,z)$, $w = h(x,z)$ has a unique solution for each choice of $(u,v,w) \in \mathbf{R}^3$ and the functions f, g and h all have continuous first partial derivatives. The Inverse Function Theorem implies that the mapping

$$\Phi(x,y,z) \quad = \quad \Big(f(x,y),\, g(y,z),\, h(x,z) \Big)$$

is a homeomorphism with a $\mathbf{C}^1$ inverse if and only if some polynomial expression in the partial derivatives of f, g and h is nonzero. Derive the explicit condition on these partial derivatives.

SOLUTION.

According to the Inverse Function Theorem, the basic condition is that the Jacobian should be nonzero at every point. In the situation being considered, the latter is given by

$$\begin{vmatrix} f_x & f_y & 0 \\ 0 & g_y & g_z \\ h_x & 0 & h_z \end{vmatrix}$$

where k_w denotes the partial derivative of k with respect to w. If we expand the determinant by the usual 3×3 rule we see that the Jacobian is equal to

$$f_x\, g_y\, h_z \;+\; f_y\, g_z\, h_x$$

and therefore the condition we want is that $f_x\, g_y\, h_z + f_y\, g_z\, h_x$ is everywhere nonzero.■

2. [*20 points*] Given a connected metric space X and a completion Y of X, show that Y is also connected.

[*5 points extra credit*] Give an example of a disconnected metric space whose completion is connected.

SOLUTIONS.

Since X is dense in its completion Y and X is connected, it follows that the closure, which is Y, is also connected.■

To answer the extra credit problem it suffices to find a complete metric space Y that is connected and a point $y \in Y$ such that $Y - \{y\}$ is dense and disconnected. There are many ways of doing this, but perhaps the simplest is to take Y to be the interval $[-1, 1]$ and $y = 0$.■

3. [*15 points*] Prove that a (finite) product of arcwise connected spaces is arcwise connected.

[*5 points extra credit*] Give complete necessary and sufficient conditions for a (finite) disjoint union of arcwise connected spaces to be arcwise connected.

SOLUTIONS.

Suppose that X_α is arcwise connected for $\alpha \in A$, and let u and v belong to $\prod_\alpha X_\alpha$. For each α let u_α and v_α be the associated coordinates of u and v respectively. Since each X_α is arcwise connected, one has continuous curves $\gamma_\alpha : [0,1] \to X_\alpha$ such that $\gamma_\alpha(0) = u_\alpha$ and $\gamma_\alpha(1) = u_\alpha$ for all α. Let $\gamma : [0,1] \to X$ be the unique continuous function whose α coordinate is given by γ_α. Then by construction we have $\gamma(0) = u$ and $\gamma(1) = v$.■

To answer the extra credit problem, note that a disjoint union of two nonempty spaces is never connected and hence never arcwise connected. Therefore the disjoint union can be arcwise connected only if there is at most one nonempty summand. Conversely, if there is one nonempty summand, then the disjoint union is homeomorphic to this arcwise connected summand and hence is arcwise conneted.■

4. [*20 points*] Recall that one definition of a quotient map in Munkres is a continuous map $f : X \to Y$ that is onto and such that for all $E \subset Y$, E is closed in $Y \iff f^{-1}(E)$ is closed in X. Show that if f is a continuous onto map from a compact space X to a Hausdorff space Y, then f is a quotient map. [*Hint:* First show that f is closed.]

SOLUTION.

The map f is closed because every continuous map from a compact space to a Hausdorff space is closed [*Proof:* E closed in $X \implies E$ is compact, which in turn implies that $f(E)$ is a compact subset of Y and hence closed because Y is Hausdorff]. To see that f is a quotient map, let $B \subset Y$ be such that $f^{-1}(B)$ is closed in X. Since f is onto we know that $B = f(f^{-1}(B))$, and since $f^{-1}(B)$ is closed in X and f is a closed mapping it also follows that B is closed in Y.∎

5. [*25 points*] Let X be a compact metric space and let $\mathcal{U}$ be a finite open covering by subsets $U_1, \cdots, U_n$. Prove that there is a finite open covering $\mathcal{V}$ of X by subsets $V_1, \cdots, V_n$ such that $\overline{V_i} \subset U_i$ for all i (informally, $\mathcal{V}$ is a *shrinking* of $\mathcal{U}$). [*Hints:* Let U_0, U_{n+1} and V_0 each denote the empty set, and inductively construct open subsets V_i such that **(1)** $\overline{V_i} \subset U_i$ for all i, **(2)** the family consisting of V_j for $j \leq i-1$ and U_j for $j \geq i$ forms an open covering of X, as follows: Let W_i be the union of all V_j's for $j \leq i-1$ and all U_j's for $j > i$, and let $F_i = X - W_i$. Explain why $F_i \subset U_i$. Since X is normal there is an open subset V_i such that $F_i \subset V_i \subset \overline{V_i} \subset U_i$. Why does the family consisting of V_j for $j \leq i$ and U_j for $j > i$ form an open covering of X?].

[*5 points extra credit*] Give a weaker condition on a compact space under which the above statement and proof are still valid (for the record, no credit will be given for simply saying that X is normal or $\mathbf{T_4}$).

SOLUTIONS.

Needless to say we shall follow the hint, which indicates an inductive construction for the open subsets V_i. We may start the induction by letting V_0 be the empty set. Suppose now that we have V_j for $j < i \leq n$ such that **(1)** $\overline{V_j} \subset U_j$ for all $j < i$, **(2)** the family consisting of V_j for $j \leq i-1$ and U_j for $j \geq i$ forms an open covering of X. As suggested in the hint, let W_i be the union of all V_j's for $j \leq i-1$ and all U_j's for $j > i$, and let $F_i = X - W_i$. The hint asks us to explain why $F_i \subset U_i$. This is true because every point in X belongs to one of the sets V_j for $j \leq i-1$ or U_j for $j \geq i$ and F_i is the set of all points that belong to the subfamily of such subsets formed by removing U_i. The only way these statements can be consistent is if every point of F_i lies in U_i. As indicated in the hint, we have an open subset V_i such that $F_i \subset V_i \subset \overline{V_i} \subset U_i$. Since the union of F_i with the open subsets forming W_i is all of X and V_i contains F_i, it follows that X is also the union of all the V_j's for $j \leq i-1$, the set V_i, and all the U_j's for $j > i$ it follows that the family consisting of V_j for $j \leq i$ and U_j for $j > i$ does form an open covering of X. By the induction hypothesis and the choice of V_i we also have $\overline{V_j} \subset U_j$ for all $j \leq i$. This completes the inductive proof of the required open subsets. When we reach $i = n$ we shall have an open covering of X with all the required properties.■

To answer the extra credit problem, it is only necessary to say that the whole argument works if X is compact Hausdorff because all such spaces are normal.■