More on second countable spaces

This is an addendum to Section VI.1 in which we describe how one can find examples of spaces that are separable but not Lindelöf, Lindelöf but not separable, and both separable and Lindelöf but not second countable. By the results of Section VI.1 it follows that such spaces examples of the first two types are neither second countable nor homeomorphic to metric spaces, and examples of the third type are likewise not homeomorphic to metric spaces.

Separability does not imply the Lindelöf property. In one of the exercises there is an example of a separable Hausdorff space X which has a closed subspace A that is both uncountable and discrete. We claim this space cannot be Lindelöf. If it were, then the closed subspace A would also be Lindelöf by another of the exercises. Since the open covering of A by one point subsets does not have a countable subcovering, it follows that A and hence X cannot be Lindelöf.∎

The Lindelöf property does not imply separability. Since compact spaces are Lindelöf, it suffices to find a compact space that is not separable. We can do this using Tychonoff's Theorem. Specifically, let $X = \{0,1\}^A$ be an uncountable product of spaces homeomorphic to $\{0,1\}$ with the discrete topology, and let A be the indexing set for the family of spaces that are factors of X. It will be convenient to assume that $|A| > 2^{\aleph_0}$. Then X is compact (by Tychonoff's Theorem) and Hausdorff (since this is true for all products of Hausdorff spaces). We claim that X is not separable.

The first thing to note is that if Y is a separable space then the cardinality of the set of continuous real valued functions on Y is precisely $2^{\aleph_0}$; the constant functions are a subset with this cardinality, and we may show there are at most $2^{\aleph_0}$ such functions as follows: If D is a countable dense subset of Y, then a continuous real valued function is completely determined by its restriction to D, and thus an upper bound for the cardinality of the set of continuous real valued functions on Y is given by the cardinality of the set of all set-theoretic real valued functions on D. But the latter cardinality is

$$\left(2^{\aleph_0}\right)^{\aleph_0} \quad = \quad 2^{\aleph_0} \cdot 2^{\aleph_0} \quad = \quad 2^{\aleph_0} \ .$$

Therefore, *if we can show that the cardinality of $X = \{0,1\}^A$ is greater than $2^{\aleph_0}$ it will follow that X is not separable.*

Given $\alpha \in A$ let f_α be given by projection onto the α factor of $\{0,1\}^A$ followed by the inclusion of $\{0,1\}$ in $\mathbf{R}$. Then $\alpha \neq \beta$ implies $f_\alpha \neq f_\beta$ because their zero sets are different, and therefore we have a subset of continuous real valued functions on $\{0,1\}^A$ that is in 1–1 correspondence with A. Since $|A| > 2^{\aleph_0}$ it follows that X cannot have a countable dense subset.∎

A separable Hausdorff space that also satisfies the Lindelöf property is not necessarily second countable. Every countable space is automatically separable and Lindelöf (the space is its own countable dense subset, and given an open covering one can extract a countable subcovering indexed by the space itself — for each x let U_x be an open subset in the covering that contains x). We also know that second countable implies first countable. Therefore it will suffice to construct a countable Hausdorff space that is not first countable.

Define an equivalence relation on the space $\mathbf{Q}^2$ of points in $\mathbf{R}^2$ with rational coordinates such that the equivalence classes are all one point sets $\{(x,y)\}$, where x and y are rational with $y \neq 0$, together with the rational points on the x-axis. This space QPP is sometimes known as the *rational pinched plane*, and by construction it is countable. We need to show that this space is Hausdorff but not first countable.

To verify that the space is Hausdorff, suppose we have two distinct points in QPP. It follows immediately that quotient space projection defines a homeomorphism from $\mathbf{Q} \times (\mathbf{Q} - \{0\})$ to the open subset $QPP - \{L\}$, where L is the equivalence class given by the x-axis. Therefore the open subset $QPP - \{L\}$ is Hausdorff, and there are disjoint open neighborhoods for each pair of distinct points in $QPP - \{L\}$. It remains to consider the case where one distinct point is L and the other comes from (x, y) where $y \neq 0$. By the definition of the quotient topology, these two points have disjoint open neighborhoods if and only if there are disjoint open subsets of $\mathbf{Q}^2$ containing (x, y) and $\mathbf{Q} \times \{0\}$. But this is true because $\mathbf{Q}^2$ is metrizable and hence $\mathbf{T_3}$.

Finally we need to show that QPP is not first countable; in fact, we claim that there is no sequence of open sets $\{ U_n \}$ in QPP such that each one contains L and every open subset W containing L contains some U_n. Let π be the quotient space projection onto QPP; by the definition of the quotient topology the open neighborhoods of L in QPP are precisely the sets of the form $\pi(V)$ where V is an open subset of $\mathbf{Q}^2$ containing the x-axis. Therefore it will suffice to show *there is an open subset* $W_0 \subset \mathbf{Q}^2$ *containing the* x*-axis such that* $\pi(W_0)$ *does not contain* **any** *of the open sets* U_n. Since U_n is the image of $\pi^{-1}(U_n)$ under π, it will suffice to show that one can find an open subset W_0 such that W_0 does not contain any of the open sets $\pi_{-1}(U_n) \subset \mathbf{Q}^2$. Note that each set $\pi^{-1}(U_n)$ is an open subset containing the rational points of the x-axis.

Given a positive integer n there is an $\varepsilon_n > 0$ such that $N_{\varepsilon_n}(\,(n, 0)\,)$ is contained in $\pi^{-1}(U_n)$. If we define W_0 to be the open set

$$\left(\bigcup_n \, (-n-1, n+1) \times (-\varepsilon_n/2, \, +\varepsilon_n/2) \right) \ \bigcap \ \mathbf{Q}^2$$

then W_0 contains the rational points of the x-axis but for each n we know that W does not contain the open segment $\{n\} \times (-\varepsilon_n, \, +\varepsilon_n)$, which is contained in $\pi^{-1}(U_n)$; it might be useful to draw a picture as an aid to understanding this. Therefore W_0 cannot contain any of the open sets $\pi^{-1}(U_n)$, and as noted above this implies that QPP is not first (or second) countable.■