EXERCISES FOR MATHEMATICS 205A

FALL 2003

The references denote sections of the texts for the course:

J. R. Munkres, *Topology* (Second Edition), Prentice-Hall, Saddle River NJ, 2000, ISBN 0–13–181629–2.

C. H. Edwards, Jr., *Advanced Calculus of Several Variables*, Dover, New York, 1994, ISBN 0–496–68336–2.

I. Foundational material

I.1 : Basic set theory

(Munkres, §§ 1, 2, 3)

Additional exercise

1. Let X be a set and let A, $B \subset X$. The *symmetric difference* $A \oplus B$ is defined by the formula

$$A \oplus B = (A - B) \ \cup \ (B - A)$$

so that $A \oplus B$ consists of all objects in A or B but not both. Prove that $\oplus$ is commutative and associative on the set of all subsets of X, that $A \oplus \emptyset = A$ for all A, that $A \oplus A = \emptyset$ for all A, and that one has the following distributivity relation for A, B, $C \subset X$:

$$A \cap (B \oplus C) \quad = \quad (A \cap B) \ \oplus \ (A \cap C)$$

[*Hint:* It might be helpful to draw some Venn diagrams.]

I.2 : Products, relations and functions

(Munkres, §§ 5, 6, 8)

Munkres, § 6, p. 44: 4(a)

Additional exercises

1.* Let X and Y be sets, suppose that A and C are subsets of X, and suppose that B and D are subsets of Y. Verify the following identities:

(*i*) $A \times (B \cap D) = (A \times B) \cap (A \times D)$
(*ii*) $A \times (B \cup D) = (A \times B) \cup (A \times D)$
(*iii*) $A \times (Y - D) = (A \times Y) - (A \times D)$
(*iv*) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
(*v*) $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$
(*vi*) $(X \times Y) - (A \times B) = (X \times (Y - B)\,) \cup (\,(X - A) \times Y)$

I.3 : Cardinal numbers

(Munkres, §§ 4, 7, 9)

Munkres, § 7, p. 51: 4**
Munkres, § 9, p. 62: 5
Munkres, § 11, p. 72: 8

Additional exercises

1.* Show that the set of *countable* subsets of $\mathbf{R}$ has the same cardinality as $\mathbf{R}$.

2. Let α and β be cardinaly numbers such that $\alpha < \beta$, and let X be a set such that $|X| = \beta$. Prove that there is a subset A of X such that $|A| = \alpha$.

I.4 : The real number system

(Munkres, § 4)

II. Metric and topological spaces

II.1 : Metrics and topologies

(Munkres, §§ 12, 13, 16, 20; Edwards, § I.8)

Munkres, § 13, p. 83: 3

Additional exercises

1.** In the integers $\mathbf{Z}$ let p be a fixed prime. For each integer $a > 0$ let

$$U_a(n) = \{\ x \mid x = n + kp^a,\ \text{some } k \in \mathbf{Z}\}\ .$$

Prove that the sets $U_a(n)$ form a basis for some topology on $\mathbf{Z}$. [*Hint:* Let $\nu_p(n)$ denote the largest nonnegative integer k such that p^k divides n and show that

$$\mathbf{d}_p(a,b) = \frac{1}{p^{\nu_p(a-b)}}$$

defines a metric on $\mathbf{Z}$].

2.* Let $A \subset X$ be closed and let $U \subset A$ be open in A. Let V be any open subset of X with $U \subset V$. Prove that $U \cup (V - A)$ is open in X.

3. Let E be a subset of the topological space X. Prove that every open subset $A \subset E$ is also open in X if and only if E itself is open in X.

II.2 : Closed sets and limit points

(Munkres, § 17)

Munkres, § 17, pp. 100–102: 2, 8(a), 8(c)*, 19*, 20*

Edwards, § I.7, pp. 48–49: 7.6

Additional exercises

0. Prove or give a counterexample to the following statement: If U and V are disjoint open subsets of a topological space X, then their closures are also disjoint.

1. Give an example to show that in a metric space the closure of an open ε disk about a point is not necessarily the set of all points whose distance from the center is $\leq \varepsilon$.

Definition. A subspace D of a topological space X is *dense* if $\overline{D} = X$; equivalently, it is dense if and only if for every nonempty open subset $U \subset X$ we have $U \cap D \neq \emptyset$.

2. For which spaces is X the only dense subset of itself?

3. Let U and V be open dense subsets of X. Prove that $U \cap V$ is dense in X.

4. A subspace A of a topological space X is said to be *locally closed* if for each $a \in A$ there is an open neighborhood U of a in X such that $U \cap A$ is closed in U. Prove that A is locally closed if and only if A is the intersection of an open subset and a closed subset.

5. (a) Suppose that D is dense in X, and let $A \subset X$. Give an example to show that $A \cap D$ is not necessarily dense in A.

(b) Suppose that $A \subset B \subset X$ and A is dense in B. Prove that A is dense in $\overline{B}$.

6. Let E be a subset of the topological space X. Prove that every closed subset $A \subset E$ is also closed in X if and only if E itself is closed in X.

7. Given a topological space X and a subset $A \subset X$, explain why the closure of the interior of A does not necessarily contain A.

8.* If U is an open subset of X and B is an arbitrary subset of X, prove that $U \cap \overline{B} \subset \overline{U \cap B}$.

9.* If X is a topological space, then the *Kuratowski closure axioms* are the following properties of the operation $A \to \mathbf{CL}(A)$ sending $A \subset X$ to its closure $\overline{A}$:

(C1) $A \subset \mathbf{CL}(A)$ for all $A \subset X$.

(C2) $\mathbf{CL}\,(\mathbf{CL}(A)\,) = \mathbf{CL}(A)$

(C3) $\mathbf{CL}(A \cup B) = \mathbf{CL}(A) \cup \mathbf{CL}(B)$ for all A, $B \subset X$.

(C4) $\mathbf{CL}(\emptyset) = \emptyset$.

Given an arbitrary set Y and a operation $\mathbf{CL}$ assigning to each subset $B \subset Y$ another subset $\mathbf{CL}(B) \subset Y$ such that **(C1) – (C4)** all hold, prove that there is a unique topology $\mathbf{T}$ on Y such that for all $B \subset Y$, the set $\mathbf{CL}(B)$ is the closure of B with respect to $\mathbf{T}$.

10. Suppose that X is a space such that $\{p\}$ is closed for all $x \in X$, andn let $A \subset X$. Prove the following statements:

(a) $\mathbf{L}(A)$ is closed in X.

(b) For each point $b \in \mathbf{L}(A)$ and open set U containing b, the intersection $U \cap A$ is infinite.

11.* Suppose that X is a set and that $\mathbf{I}$ is an operation on subsets of X such that the following hold:

(i) $\mathbf{I}(X) = X$.
(ii) $\mathbf{I}(A) \subset A$ for all $A \subset X$.
(iii) $\mathbf{I}(\mathbf{I}(A)) = \mathbf{I}(A)$ for all $A \subset X$.
(iv) $\mathbf{I}(A \cap B) = \mathbf{I}(A) \cap \mathbf{I}(B)$.

Prove that there is a unique topology $\mathbf{T}$ on X such that $U \in \mathbf{T}$ if and only if $\mathbf{I}(A) = A$.

12.* If X is a topological space and $A \subset X$ then the *exterior* of X, denoted by $\mathrm{Ext}(X)$, is defined to be $X - \overline{A}$. Prove that this construction has the following properties:

(a) $\mathrm{Ext}(A \cup B) = \mathrm{Ext}(A) \cap \mathrm{Ext}(B)$.
(b) $\mathrm{Ext}(A) \cap A = \emptyset$.
(c) $\mathrm{Ext}(\emptyset) = X$.
(d) $\mathrm{Ext}(A) \subset \mathrm{Ext}\,(\mathrm{Ext}\,(\mathbf{L}(A)\,)\,)$.

13. Let A_1 and A_2 be subsets of a topological space X, and let B be a subset of $A_1 \cap A_2$ that is closed in both A_1 and A_2 with respect to the subspace to topologies on each of these sets. Prove that B is closed in $A_1 \cup A_2$.

14.* Suppose that A is a closed subset of a topological space X and B is the closure of $\mathrm{Int}(A)$. Prove that $\overline{B} \subset A$ and $\mathrm{Int}(B) = \mathrm{Int}(A)$.

15. Let X be a topological space and let $A \subset Y \subset X$.

(a) Prove that the interior of A with respect to X is contained in the interior of A with respect to Y.

(b)* Prove that the boundary of A with respect to Y is contained in the intersection of Y with the boundary of A with respect to X.

(c)* Give examples to show that the inclusions in the preceding two statements may be proper (it suffices to give one example for which both inclusions are proper).

II.3 : Continuous functions

(Munkres, §§ 18, 21; Edwards, § I.7)

Munkres, § 18, pp. 111–112: 2, 6*, 8(a) with $Y = \mathbf{R}$, 9(c)

Edwards, § I.7, p. 48: 7.3, 7.4

Additional exercises

1. Give examples of continuous functions from $\mathbf{R}$ to itself that are neither open nor closed.

2. Let X be a topological space, and let $f, g : X \to \mathbf{R}$ be continuous. Prove that the functions $|f|$, $\max(f, g)$ [whose value at $x \in X$ is the larger of $f(x)$ and $g(x)$] and $\min(f, g)$ [whose value at $x \in X$ is the smaller of $f(x)$ and $g(x)$] are all continuous. [*Hints:* If $h : X \to \mathbf{R}$ is continuous, what can one say about the sets of points where $h = 0$, $h < 0$ and $h > 0$? What happens if we take $h = f - g$?]

3.* Let $f : X \to Y$ be a set-theoretic mapping of topological spaces.

(a) Prove that f is open if and only if $f(\,\mathrm{Int}(A)\,) \subset \mathrm{Int}(f(A))$ for all $A \subset X$ and that f is closed if and only if $\overline{f(A)} \subset f(\overline{A})$ for all $A \subset X$.

(b) Using this and other results from the course notes, prove that f is continuous and closed if and only if $\overline{f(A)} = f(\overline{A})$ for all $A \subset X$ and f is is continuous and open if and only if $f^{-1}(\,\mathrm{Int}(B)\,) \subset \mathrm{Int}(f(B))$ for all $B \subset Y$.

II.4 : Cartesian products

(Munkres, §§ 15, 19)

Munkres, § 18, pp. 111–112: 10, 11

Additional exercises

1.** ("A product of products is a product,") Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of nonempty sets, and let $\mathcal{A} = \cup\{\mathcal{A}_\beta \mid \beta \in \mathcal{B}\}$ be a partition of $\mathcal{A}$. Construct a bijective map of $\prod\{A_\alpha \mid \alpha \in \mathcal{A}\}$ to the set

$$\prod_\beta \{\prod\{A_\alpha \mid \alpha \in \mathcal{A}_\beta\}\ \}\ .$$

If each A_α is a topological space and we are working with product topologies, prove that this bijection is a homeomorphism.

2.** *Non-Hausdorff topology.* In topology courses one is ultimately interested in spaces that are Hausdorff. However, there are contexts in which certain types of non-Hausdorff spaces arise (for example, the Zariski topologies in algebraic geometry, which are defined in most textbooks on that subject).

(a) A topological space X is said to be *irreducible* if it cannot be written as a union $X = A \cup B$, where A and B are proper closed subspaces of X. Show that X is irreducible if and only if every pair of nonempty open subsets has a nonempty intersection. Using this, show that an open subset of an irreducible space is irreducible.

(b) Show that every set with the indiscrete topology is irreducible, and every infinite set with the finite complement topology is irreducible.

(c) Show that an irreducible Hausdorff space contains at most one point.

3. Let $f : X \to Y$ be a map, and define the *graph* of f to be the set Γ_f of all points $(x, y) \in X \times Y$ such that $y = f(x)$. Prove that the map $x \to (x, f(x))$ is a homeomorphism from X to Γ_f if and only if f is continuous.

4. Let X be a topological space that is a union of two closed subspaces A and B, where each of A and B is Hausdorff in the subspace topology. Prove that X is Hausdorff.

5.* Let A be some nonempty set, let $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ be families of topological spaces, and for each $\alpha \in A$ suppose that $f_\alpha : X_\alpha \to Y_\alpha$ is a homeomorphism. Prove that the product map

$$\prod_\alpha f_\alpha \ :\ \prod_\alpha X_\alpha \longrightarrow \prod_\alpha Y_\alpha$$

is also a homeomorphism. [*Hint:* What happens when you take the product of the inverse maps?]

6. Let X be a topological space and let $T : X \times X \times X \to X \times X \times X$ be the map that cyclically permutes the coordinates: $T(x, y, z) = (z, x, y)$ Prove that T is a homeomorphism. [*Hint:* What is the test for continuity of a map into a product? Can you write down an explicit formula for the inverse function?]

III. Spaces with special properties

III.1 : Compact spaces – I

(Munkres, §§ 26, 27)

Munkres, § 26, pp. 170–172: 3, 7*, 8*
Munkres, § 27, pp. 177–178: 2(a) (b) (c) (d) (e)*

Additional exercises

1. Let X be a compact Hausdorff space, and let $f : X \to X$ be continuous. Define $X_1 = X$ and $X_{n+1} = f(X_n)$, and set $A = \bigcap_n X_n$. Prove that A is a nonempty subset of X and $f(A) = A$.

2.** A topological space X is said to be a k-**space** if it satisfies the following condition: *A subset $A \subset X$ is closed if and only if for all compact subsets $K \subset X$, the intersection $A \cap K$ is closed.* It turns out that a large number of the topological spaces one encounters in topology, geometry and analysis are k-spaces (including all metric spaces and compact Hausdorff spaces), and the textbooks by Kelley and Dugundji contain a great deal of information about these k-spaces (another important reference is the following paper by N. E. Steenrod: *A convenient category of topological spaces*, Michigan Mathematical Journal **14** (1967), pp. 133–152).

(a) Prove that if $(X, \mathbf{T})$ is a Hausdorff topological space then there is a unique minimal topology $\mathbf{T}^\kappa$ containing $\mathbf{T}$ such that $\mathcal{K}(X) = (X, \mathbf{T}^\kappa)$ is a Hausdorff k-space.

(b) Prove that if $f : X \to Y$ is a continuous map of Hausdorff topological spaces, then f is also continuous when viewed as a map from $\mathcal{K}(X) \to \mathcal{K}(Y)$.

3.** *Non-Hausdorff topology revisited.* A topological space X is *noetherian* if every nonempty family of open subsets has a maximal element. This class of spaces is also of interest in algebraic geometry.

(a) (*Ascending Chain Condition*) Show that a space X is noetherian if and only if every increasing sequence of open subsets

$$U_1 \subset U_2 \subset \ldots$$

stabilizes; *i.e.*, there is some positive integer N such that $n \geq N$ implies $U_n = U_N$.

(b) Show that a space X is noetherian if and only if every open subset is compact.

(c) Show that a noetherian Hausdorff space is finite (with the discrete topology). [*Hint:* Show that every open subset is closed.]

(d) Show that a subspace of a noetherian space is noetherian.

III.2 : Complete metric spaces

(Munkres, §§ 43, 45)

Munkres, § 43, pp. 270–271: 1, 3(b), 6(a)(c)*

Additional exercises

1. Show that the Nested Intersection Property for complete metric spaces does not necessarily hold for nested sequences of closed subsets $\{ A_n \}$ if $\lim_{n\to\infty} \mathrm{diam}(A_n) \neq 0$; *i.e.*, in such cases one might have $\cap_n A_n = \emptyset$. [*Hint:* Consider the set A_n of all continuous functions f from $[0,1]$ to itself that are zero on $[\frac{1}{n}, 1]$ and also satisfy $f(0) = 1$.]

2.** Let ℓ^2 be the set of all real sequences $\mathbf{x} = \{ x_n \}$ such that $\sum_n |x_n|^2$ converges. The results of Exercise 10 on page 128 of Munkres show that ℓ^2 is a normed vector space with the norm

$$| \mathbf{x} | = \left(\sum_n |x_n|^2 \right)^{1/2} .$$

Prove that ℓ^2 is complete with respect to the associated metric. [*Hint:* If $\mathbf{p}_i$ gives the i^{th} term of an element in ℓ^2, show that $\mathbf{p}_i$ takes Cauchy sequences to Cauchy sequences. This gives a candidate for the limit of a Cauchy sequence in ℓ^2. Show that this limit candidate actually lies in ℓ^2 and that the Cauchy sequence converges to it. See also Royden, *Real Analysis*, Section 6.3, pages 123–127, and also Rudin, *Principles of Mathematical Analysis*, Theorem 11.42 on page 329–330 together with the discussion on the following two pages.]

III.3 : Implications of completeness

(Munkres, § 48; Edwards, § III.1)

Munkres, § 27, pp. 177–178: 6*
Munkres, § 48, pp. 298–300: 1, 2*, 4*

Edwards, § III.1, p. 171: 1.6, 1.7*

Additional exercises

1. Let A and B be subspaces of X and Y respectively such that $A \times B$ is nowhere dense in $X \times Y$ (with respect to the product topology). Prove that either A is nowhere dense in X or B is nowhere dense in Y, and give an example to show that "or" cannot be replaced by "and."

2. Is there an uncountable topological space of the first category?

3.* Let X be a metric space. A map $f : X \to X$ is said to be an *expanding similarity* of X if f is onto and there is a constant $C > 1$ such that

$$\mathbf{d}\,(\,f(u),\, f(v)\,) \;=\; C \cdot \mathbf{d}(u, v)$$

for all u, $v \in X$ (hence f is 1–1 and uniformly continuous). Prove that every expanding similarity of a complete metric space has a unique fixed point. [*Hint and comment:* Why does f have an inverse that is uniformly continuous, and why does $f(x) = x$ hold if and only if $f^{-1}(x) = x$? If $X = \mathbf{R}^n$ and a similarity is given by $f(x) = cAx + b$ where A comes from an orthogonal matrix and either $0 < c < 1$ or $c > 1$, then one can prove the existence of a unique fixed point directly using linear algebra.]

III.4 : Connected spaces

(Munkres, §§ 23, 24, 25)

Munkres, § 23, p. 152: 3, 4, 5, 9, 12* assuming Y is closed
Munkres, § 24, pp. 157–159: 1(b)

Additional exercises

1. Prove that a topological space X is connected if and only if every open covering $\mathcal{U} = \{U_\alpha\}$ has the following property: For each pair of sets U, V in $\mathcal{U}$ there is a sequence of open sets $\{U_0, U_1, ...U_n\}$ in $\mathcal{U}$ such that $U = U_0$, $V = U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for all i.

2. Let X be a connected space, and let $\mathcal{R}$ be an equivalence relation that is locally constant (for each point x all points in some neighborhood of x lie in the $\mathcal{R}$-equivalence class of x). Prove that $\mathcal{R}$ has exactly one equivalence class.

3. Prove that an open subset in $\mathbf{R}^n$ can have at most countably many components. Give an example to show this is not necessarily true for closed sets.

III.5 : Variants of connectedness

(Munkres, §§ 23, 24, 25)

Munkres, § 24, pp. 157–159: 8(b)
Munkres, § 25, pp. 162–163: 10(a) (b) (c)* only Examples A and B

Additional exercises

1. Prove that a compact locally connected space has only finitely many components.

2.* Give and example to show that if X and Y are locally connected metric spaces and $f : X \to Y$ is continuous then $f(X)$ is not necessarily locally connected.

IV. Smooth mappings

IV.1 : Linear approximations

(Edwards, §§ II.1, II.2)

Edwards, §§ II.1, p. 75: 2.6*, 2.8*
Edwards, §§ II.2, pp. 88–90: 3.3*, 3.16*

IV.2 : Properties of smooth functions

(Edwards, § II.3)

Edwards, § II.3, pp. 88–90: 3.4*, 3.11, 3.12*, 3.13*, 3.14
Edwards, §§ III.2, p. 180: 2.1*, 2.8**

Additional exercises

1. (a) Suppose that X and Y are subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively and that $f : X \to Y$ and $g : Y \to \mathbf{R}^p$ are maps that satisfy Lipschitz conditions. Prove that the composite $g \circ f$ also satisfies a Lipschitz condition.

(b) Suppose that $X \subset \mathbf{R}^n$, and let $f,\ g : X \to \mathbf{R}^m$ and $h : X \to \mathbf{R}$ satisfy Lipschitz conditions. Prove that $f + g$ satisfies a Lipschitz condition and if X is compact then $h \cdot f$ also satisfies a Lipschitz condition. If $h > 0$ and X is compact, does $1/h$ satisfy a Lipschitz condition? Prove this or give a counterexample.

(c) Suppose that $X \subset \mathbf{R}^n$, and let $f : X \to \mathbf{R}^m$ be given. Prove that f satisfies a Lipschitz condition if and only if all of its coordinate functions do.

2.* In the notation of the preceding exercise, suppose that $X = A \cup B$ and that f is continuous and satisfies Lipschitz conditions on A and B as well as on an open neighborhood of $A \cap B$. Does f satisfy a Lipschitz condition on $A \cup B$? Prove this or give a counterexample. What happens if we assume A and B are compact? Justify your answer.

IV.3 : Inverse Function Theorem

(Edwards, §§ III.2, III.3)

Edwards, §§ III.3, pp. 194–196: 3.1, 3.3*, 3.4*, 3.14*

Additional exercises

0. Prove that $F(x, y) = (e^x + y,\ x - y)$ defines a $\mathbf{C}^\infty$ homeomorphism of $\mathbf{R}^2$ with a $\mathbf{C}^\infty$ inverse.

1. Prove that $F(x, y) = (xe^y + y,\ xe^y - y)$ defines a $\mathbf{C}^\infty$ homeomorphism of $\mathbf{R}^2$ with a $\mathbf{C}^\infty$ inverse.

2.* Prove that

$$F(x, y, z) = \left(\frac{x}{2 + y^2} + ye^z, \frac{x}{2 + y^2} - ye^z, 2ye^z + z \right)$$

defines a $\mathbf{C}^\infty$ homeomorphism of $\mathbf{R}^3$ with a $\mathbf{C}^\infty$ inverse.

3. Let $f(x, y) = (x + y, x^2 + y)$. Check that f meets the conditions to have a local inverse near $f(1, 0) = (1, 1)$, and if g is this local inverse find $Dg(1, 1)$ without finding a formula for the inverse function explicitly.

4.* Consider the mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by $f(x, y) = (x^2 + y^2, 2xy)$. Show that the Jacobian vanishes on the lines $y = \pm x$. What is the image of f? [*Hint:* Try using polar coordinates.] The Inverse Function Theorem guarantees that f has a local inverse at $f(1, 0) = (1, 0)$. Find the inverse explicitly and describe a region on which it is defined.

5. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $f(v, w) = (v + w + e^{vw}, w + 2v + e^{-vw})$. Prove that there is a local inverse at $f(0, 0) = (1, 1)$ and if

$$g(x, y) = (\, v(x, y)\,,\ w(x, y)\,)$$

show that

$$\frac{\partial v}{\partial x}(1,1) + \frac{\partial w}{\partial x}(1,1) = \frac{\partial v}{\partial y}(1,1) \ .$$

V. Constructions on spaces

V.1 : Quotient spaces

(Munkres, § 22)

Munkres, § 22, pp. 144–145: 4

Additional exercises

0. Suppose that X is a space with the discrete topology and $\mathcal{R}$ is an equivalence relation on X. Prove that the quotient topology on $X/\mathcal{R}$ is discrete.

1.* If A is a subspace of X, a continuous map $r : X \to A$ is called a *retraction* if the restriction of r to A is the identity. Show that a retraction is a quotient map.

2. Let $\mathcal{R}$ be an equivalence relation on a space X, and assume that $A \subset X$ contains points from every equivalence class of $\mathcal{R}$. Let $\mathcal{R}_0$ be the induced equivalence relation on A, and let

$$j : A/\mathcal{R}_0 \to X/\mathcal{R}$$

be the associated $1-1$ correspondence of equivalence classes. Prove that j is a homeomorphism if there is a retraction $r : X \to A$ such that each set $r^{-1}(\{a\})$ is contained in an $\mathcal{R}$-equivalence

3. (a) Let 0 denote the origin in $\mathbf{R}^3$. In $\mathbf{R}^3 - \{0\}$ define $x\mathcal{R}y$ if y is a nonzero multiple of x (geometrically, if x and y lie on a line through the origin). Show that $\mathcal{R}$ is an equivalence relation; the quotient space is called the *real projective plane* and denoted by $\mathbf{RP}^2$.

(b) Using the previous exercise show that $\mathbf{RP}^2$ can also be viewed as the quotient of S^2 modulo the equivalence relation $x \sim y \Longleftrightarrow y = \pm x$. In particular, this shows that $\mathbf{RP}^2$ is compact. [*Hint:* Let r be the radial compression map that sends v to $|v|^{-1}v$.]

4.** In $D^2 = \{x \in \mathbf{R}^2 |\ |x| \leq 1\}$, consider the equivalence relation generated by the condition $x\mathcal{R}'y$ if $|x| = |y| = 1$ and $y = -x$. Show that this quotient space is homeomorphic to $\mathbf{RP}^2$.

[*Hints:* Use the description of $\mathbf{RP}^2$ as a quotient space of S^2 from the previous exercise, and let $h : D^2 \to S^2$ be defined by

$$h(x,y) = (x, y, \sqrt{1 - x^2 - y^2}\ ).$$

Verify that h preserves equivalence classes and therefore induces a continuous map $\overline{h}$ on quotient spaces. Why is $\overline{h}$ a $1-1$ and onto mapping? Finally, prove that $\mathbf{RP}^2$ is Hausdorff and $\overline{h}$ is a closed mapping.]

5. Suppose that X is a topological space with topology $\mathbf{T}$, and suppose also that Y and Z are sets with set-theoretic maps $f : X \to Y$ and $g : Y \to Z$. Prove that the quotient topologies satisfy the condition

$$(g \circ f)_* \mathbf{T} = g_* \left(f_* \mathbf{T} \right) \ .$$

(Informally, a quotient of a quotient is a quotient.)

6.* If Y is a topological space with a topology $\mathbf{T}$ and $f ; X \to Y$ is a set-theoretic map, then the *induced topology* $f^* \mathbf{T}$ on X is defined to be the set of all subsets $W \subset X$ having the form $f^{-1}(U)$ for some open set $U \in \mathbf{T}$. Prove that $f^* \mathbf{T}$ defines a topology on X, that it is the unique smallest topology on X for which f is continuous, and that if $h : Z \to X$ is another set-theoretic map then

$$(f \circ h)^* \mathbf{T} = h^* \left(f^* \mathbf{T} \right) \ .$$

7. Let X and Y be topological spaces, and define an equivalence relation $\mathcal{R}$ on $X \times Y$ by $(x, y) \sim (x', y')$ if and only if $x = x'$. Show that $X \times Y / \mathcal{R}$ is homeomorphic to X.

8. Let $\mathcal{R}$ be an equivalence relation on a topological space X, let $\Gamma_{\mathcal{R}}$ be the graph of $\mathcal{R}$, and let $\pi : X \to X/\mathcal{R}$ be the quotient projection. Prove the following statements:
(a) If $X/\mathcal{R}$ satisfies the Hausdorff Separation Property then $\Gamma_{\mathcal{R}}$ is closed in $X \times X$.
(b) If $\Gamma_{\mathcal{R}}$ is closed and π is open, then $X/\mathcal{R}$ is Hausdorff.
(c) If $\Gamma_{\mathcal{R}}$ is open then $X/\mathcal{R}$ is discrete.

V.2 : Sums and cutting and pasting

(not covered in the texts)

Additional exercises

1. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Prove that X is locally connected if and only if each A_α is locally connected.

2. In the preceding exercise, formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be compact.

3.** Prove that $\mathbf{RP}^2$ can be constructed by identifying the edge of a Möbius strip with the edge circle on a closed 2-dimensional disk by filling in the details of the following argument: Let $A \subset S^2$ be the set of all points $(x, y, z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T(A) = A$ and $T(B) = B$ so that each of A and B (as well as their intersection) can be viewed as a union of equivalence classes for the equivalence relation that produces $\mathbf{RP}^2$. By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where $\mathrm{sign}(z) = \pm 1$, and thus it follows that the image of B in the quotient space is homeomorphic to $B_+ \cong D^2$. Now consider A. There is a homeomorphism h from $S^1 \times [-1, 1]$ to A sending (x, y, t) to $(\alpha(t) x, \alpha(t) y, \frac{1}{2} t)$ where

$$\alpha(t) = \sqrt{1 - \frac{t^2}{4}}$$

and by construction $h(-v) = -h(v)$. The image of A in the quotient space is thus the quotient of $S^1 \times [-1, 1]$ modulo the equivalence relation $u \sim v \iff u = \pm v$. This quotient space is in turn homeomorphic to the quotient space of the upper semicircular arc S^1_+ (all points with nonnegative y-coordinate) modulo the equivalence relation generated by setting $(-1, 0, t)$ equivalent to $(1, 0, -t)$, which yields the Möbius strip. The intersection of this subset in the quotient with the image of B

is just the image of the closed curve on the edge of B_+, which also represents the edge curve on the Möbius strip.

4. Suppose that the topological space X is a union of two closed subspaces A and B, let $C = A \cap B$, let $h : C \to C$ be a homeomorphism, and let $A \cup_h B$ be the space formed from $A \sqcup B$ by indentifying $x \in C \subset A$ with $h(x) \in C \subset B$. Prove that $A \cup_h B$ is homeomorphic to X if h extenhds to a homeomorphism $H : A \to A$, and give an example for which X is not homeomorphic to $A \cup_h B$. [*Hint:* Construct the homeomorphism using H in the first case, and consider also the case where $X = S^1 \sqcup S^1$, with $A_\pm == S^1_\pm \sqcup S^1_\pm$; then $C = \{\pm 1\} \times \{1, 2\}$, and there is a homeomorphism from h to itself such that $A_+ \cup_h A_-$ is connected.]

5.** *One-point unions.* One conceptual problem with the disjoint union of topological spaces is that it is never connected except for the trivial case of one summand. In many geometrical and topological contexts it is extremely useful to construct a modified version of disjoint unions that is connected if all the pieces are. Usually some additional structure is needed in order to make such constructions.

In this exercise we shall describe such a construction for objects known as *pointed spaces* that are indispensable for many purposes (*e.g.*, the definition of fundamental groups as in Munkres). A pointed space is a pair (X, x) consisting of a topological space X and a point $x \in X$; we often call x the *base point*, and unless stated otherwise *the one point set consisting of the base point is assumed to be closed.* If (Y, y) is another pointed space and $f : X \to Y$ is continuous, we shall say that *f is a base point preserving continuous map from (X, x) to (Y, y)* if $f(x) = y$, In this case we shall often write $f : (X, x) \to (Y, y)$. Identity maps are base point preserving, and composites of base point preserving maps are also base point preserving.

(a) Given a finite collection of pointed spaces (X_i, x_i), define an equivalence relation on $\coprod_i X_i$ whose equivalence classes consist of $\coprod_j \{x_j\}$ and all one point sets y such that $y \notin \coprod_j \{x_j\}$. Define the *one point union* or *wedge*

$$\bigvee_{i=1}^{n} (X_j, x_j) = (X_1, x_1) \vee \cdots \vee (X_n, x_n)$$

to be the quotient space of this equivalence relation with the quotient topology. The base point of this space is taken to be the class of $\coprod_j \{x_j\}$.

(a) Prove that the wedge is a union of closed subspaces Y_j such that each Y_j is homeomorphic to X_j and if $j \neq k$ then $Y_j \cap Y_k$ is the base point. Explain why $\vee_k\ (X_k, x_k)$ is Hausdorff if and only if each X_j is Hausdorff, why $\vee_k\ (X_k, x_k)$ is compact if and only if each X_j is compact, and why $\vee_k\ (X_k, x_k)$ is connected if and only if each X_j is connected (and the same holds for arcwise connectedness).

(b) Let $\varphi_j : (X_j, x_j) \to \vee_k\ (X_k, x_k)$ be the composite of the injection $X_j \to \coprod_k X_k$ with the quotient projection; by construction φ_j is base point preserving. Suppose that (Y, y) is some arbitrary pointed space and we are given a sequence of base point preserving continuous maps $F_j : (X_j, x_j) \to (Y, y)$. Prove that there is a unique base point preserving continuous mapping

$$F : \vee_k\ (X_k, x_k) \to (Y, y)$$

such that $F \circ \varphi_j = F_j$ for all j.

(c) In the infinite case one can carry out the set-theoretic construction as above but some care is needed in defining the topology. Show that if each X_j is Hausdorff and one takes the so-called *weak topology* whose closed subsets are generated by the family of subsets $\varphi_j(F)$ where F is closed in X_j for some j, then [1] a function h from the wedge into some other space Y is continuous if and only if each composite $h \circ \varphi_j$ is continuous, [2] the existence and uniqueness theorem for mappings from the wedge (in the previous portion of the exercise) generalizes to infinite wedges with the so-called weak topologies.

(d) Suppose that we are given an infinite wedge such that each summand is Hausdorff and contains at least two points. Prove that the wedge with the so-called weak topology is not compact.

Remark. If each of the summands in (d) is compact Hausdorff, then there is a natural candidate for a *strong topology* on a countably infinite wedge which makes the latter into a compact Hausdorff space. In some cases this topology can be viewed more geometrically; for example, if each (X_j, x_j) is equal to $(S^1, 1)$ and there are countably infinitely many of them, then the space one obtains is the *Hawaiian earring* in $\mathbf{R}^2$ given by the union of the circles defined by the equations

$$\left(x - \frac{1}{2^k}\right)^2 + y^2 = \frac{1}{2^{2k}} \ .$$

As usual, drawing a picture may be helpful. The k^{th} circle has center $(1/2^k,\ 0)$ and passes through the origin; the y-axis is the tangent line to each circle at the origin.

VI. Spaces with additional properties

VI.1 : Second countable spaces

(Munkres, § 30)

Munkres, § 30, pp. 194–195: 9 (first part only), 10, 13*, 14*

Additional exercises

1. If $(X, \mathbf{T})$ is a second countable Hausdorff space, prove that the cardinalities of both X and $\mathbf{T}$ are less than or equal to $2^{\aleph_0}$. (Using the formulas for cardinal numbers in Section I.3 of the course notes and the separability of X one can prove a similar inequality for $\mathbf{BC}(X)$.)

2.** *Separability and subspaces.* The following example shows that a closed subspace of a separable Hausdorff space is not necessarily separable.

(a) Let X be the upper half plane $\mathbf{R} \times [0, \infty)$ and take the topology generated by the usual metric topology plus the following sets:

$$T_\varepsilon(x) = \{\,(x,0)\,\} \quad \cup \quad N_\varepsilon\left(\,(x,\varepsilon)\,\right), \quad \text{where} \quad x \in \mathbf{R} \quad \text{and} \quad \varepsilon \ > \ 0$$

Geometrically, one takes the interior region of the circle in the upper half plane that is tangent to the x-axis at $(x, 0)$ and adds the point of tangency. — Show that the x-axis is a closed subset and has the discrete topology.

(b) Explain why the space in question is Hausdorff. [*Hint:* The topology contains the metric topology. If a topological space is Hausdorff and we take a larger topology, why is the new topology Hausdorff?]

(c) Show that the set of points (u, v) in X with $v > 0$ and $u, v \in \mathbf{Q}$ is dense. [*Hint:* Reduce this to showing that one can find such a point in every set of the form $T_\varepsilon(x)$.]

3. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be second countable, separable or Lindelöf.

VI.2 : Compact spaces – II

(Munkres, §§ 26, 27,28)

Munkres, § 28, pp. 181–182: 6

Additional exercises

1.* Let X be a compact Hausdorff space, let Y be a Hausdorff space, and let $f : X \to Y$ be a continuous map such that f is locally 1–1 (each point x has a neighborhood U_x such that $f|U_x$ is 1–1) and there is a closed subset $A \subset X$ such that $f|A$ is 1–1. Prove that there is an open neighborhood V of A such that $f|V$ is 1–1. [*Hint:* A map g is 1–1 on a subset B is and only if

$$B \times B \cap (g \times g)^{-1}(\Delta_Y) = \Delta_B,$$

where Δ_S denotes the diagonal in $S \times S$. In the setting of the exercise show that

$$(f \times f)^{-1}(\Delta_Y) = \Delta_X \cup D',$$

where D' is closed and disjoint from the diagonal. Also show that the subsets D' and $A \times A$ are disjoint, and find a square neighborhood of $A \times A$ disjoint from D'.]

2. Let U be open in $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^n$ be a $\mathbf{C}^1$ map such that $Df(x)$ is invertible for all $x \in U$ and there is a compact subset $A \subset U$ such that $f|A$ is 1–1. Prove that there is an open neighborhood V of A such that $f|V$ is a homeomorphism onto its image.

3.** Let $\mathbf{d}_p$ be the metric on the integers constructed in Exercise I.1.1, and let $\widehat{\mathbf{Z}_p}$ be the completion of this metric space. Prove that $\widehat{\mathbf{Z}_p}$ is (sequentially) compact. [*Hint:* For each integer $r > 0$ show that every integer is within p^{-r} of one of the first p^{r+1} nonnegative integers. Furthermore, each open neighborhood of radius p^{-r} centered at one of these integers a is a union of p neighborhoods of radius $p^{-(r+1)}$ over all of the first p^{r+2} integers b such that $b \equiv a$ mod p^{r+1}. Now let $\{ a_n \}$ be an infinite sequence of integers, and assume the sequence takes infinitely many distinct values (otherwise the sequence obviously has a convergent subsequence). Find a sequence of positive integers $\{ b_r \}$ such that the open neighborhood of radius p^{-r} centered at b_r contains infinitely many points in the sequence and $b_{r+1} \equiv b_r$ mod p^{r+1}. Prove that this yields a Cauchy sequence in $\mathbf{Z}$ (with respect to the $\mathbf{d}_p$ metric). Form a subsequence of $\{ a_n \}$ by choosing distinct points $a_{n(k)}$ recursively such that $n(k) > n(k-1)$ and $a_{n(k)} \in N_{p^{-k}}(b_k)$. Prove that this subsequence is a Cauchy sequence and hence converges.]

VI.3 : Separation axioms

(Munkres, §§ 31, 32, 33, 35)

Munkres, § 26, pp. 170–172: 11*
Munkres, § 33, pp. 212–214: 2(a) (for metric spaces), 6(a), 8

Additional exercises

1. If $(X, \mathbf{T})$ is compact Hausdorff and $\mathbf{T}^*$ is strictly contained in $\mathbf{T}$, prove that $(X, \mathbf{T}^*)$ is compact but not Hausdorff.

2. (a) Prove that a topological space is $\mathbf{T_3}$ if and only if it is $\mathbf{T_1}$ and there is a basis $\mathcal{B}$ such that for every $x \in X$ and every open set $V \in \mathcal{B}$ containing x, there is an open subset $W \in \mathcal{B}$ such that $x \in W \subset \overline{W} \subset V$.

(b)** Prove that the space constructed in Exercise VI.1.2 is $\mathbf{T_3}$. [*Hint:* Remember that the "new" topology contains the usual metric topology.]

3. If X is a topological space and $A \subset X$ is nonempty then X/A (in words, "X mod A" or "X modulo A collapsed to a point") is the quotient space whose equivalence classes are A and all one point subsets $\{x\}$ such that $x \notin A$. Geometrically, one is collapsing A to a single point.

(a) Suppose that A is closed in X. Prove that X/A is Hausdorff if either X is compact Hausdorff or X is metric (in fact, if X is $\mathbf{T_4}$).

(b) Still assuming A is closed but not making any assumptions on X (except that it be nonempty), show that the quotient map $X \to X/A$ is always closed but not necessarily open. [*Note:* For reasons that we shall not discuss, it is appropriate to define $X/\emptyset$ to be the disjoint union $X \sqcup \{\emptyset\}$.]

VI.4 : Local compactness and compactifications

(Munkres, §§ 29, 37, 38)

Munkres, § 38, pp. 241–242: 2*, 3 (just give a necessary condition on the topology of the space)

Additional exercises

Definition. If $f : X \to Y$ is continuous, then f is *proper* (or *perfect*) if for each compact subset $K \subset Y$ the inverse image $f^{-1}(K)$ is a compact subset of X.

1. Suppose that $f : X \to Y$ is a continuous map of noncompact locally compact T_2 spaces. Let $f^\bullet : X^\bullet \to Y^\bullet$ be the map of one point comapactifications defined by $f^\bullet|X = f$ and $f^\bullet(\infty_X) = (\infty_Y)$. Prove that f is proper if and only if $f^\bullet$ is continuous.

2. Prove that a proper map of noncompact locally compact Hausdorff spaces is closed.

3. If $\mathbf{F}$ is the reals or complex numbers, prove that every polynomial map $p : \mathbf{F} \to \mathbf{F}$ is proper. [*Hint:* Show that

$$\lim_{|x| \to \infty} \big|p(z)\big| \quad = \quad \infty$$

and use the characterization of compact subsets as closed and bounded subsets of $\mathbf{F}$.]

4.* Let ℓ^2 be the complete metric space described above, and view $\mathbf{R}^n$ as the subspace of all sequences with $x_k = 0$ for $k > n$. Let $A_n \subset \ell^2 \times \mathbf{R}$ be the set of all ordered pairs (x, t) with $x \in \mathbf{R}^n$ and $0 < t \leq 2^{-n}$. Show that $A = \bigcup_n A_n$ is locally compact but its closure is not. Explain why this shows that the completion of a locally compact metric space is not neccesarily locally compact. [*Hint:* The family $\{ A_n \}$ is a locally finite family of closed locally compact subspaces in A. Use this to show that the union is locally compact, and show that the closure of A contains all of $\ell^2 \times \{0\}$. Explain why ℓ^2 is not locally compact.]

5. Let X be a compact Hausdorff space, and let $U \subset X$ be open and noncompact. Prove that the collapsing map $c : X \to U^\bullet$ such that $c|U = \mathrm{id}_U$ and $c = \infty_U$ on $X - U$ is continuous. Show also that c is not necessarily open.

6. (a) Explain why a compact Hausdorff space has no nontrivial Hausdorff abstract closures.

(b) Prove that a Hausdorff space X has a maximal abstract Hausdorff closure that is unique up to equivalence. [*Hint:* Consider the identity map.]

7. Suppose that X is compact Hausdorff and A is a closed subset of X. Prove that X/A is homeomorphic to the one point compactification of $X - A$.

VI.5 : Metrization theorems

(Munkres, §§ 39, 40, 41, 42)

Munkres, § 40, p. 252: 2, 3

Additional exercises

1. A *pseudometric space* is a pair $(X, \mathbf{d})$ consisting of a nonempty set X and a function $\mathbf{d}X \times X \to \mathbf{R}$ that has all the properties of a metric except possibly the property that $\mathbf{d}(u, v) = 0$.

(a) If ε-neighborhoods and open sets are defined as for metric spaces, explain why one still obtains a topology for pseudometric spaces.

(b) Given a pseudometric space, define a binary relation $x \sim y$ if and only if $\mathbf{d}(x, y) = 0$. Show that this defines an equivalence relation and that $\mathbf{d}(x, y)$ only depends upon the equivalence classes of x and y.

(c) Given a sequence of pseudometrics $\mathbf{d}_n$ on a set X, let $\mathbf{T}_\infty$ be the topology generated by the union of the sequence of topologies associated to these pseudometrics, and suppose that for each pair of distinct points $u\, v \in X$ there is some n such that $\mathbf{d}_n(u, v) > 0$. Prove that $(X, \mathbf{T}_\infty)$ is metrizable and that

$$\mathbf{d}_\infty \quad = \quad \sum_{n=1}^{\infty} \frac{\mathbf{d}_n}{2^n(1 + \mathbf{d}_n)}$$

defines a metric whose underlying topology is $\mathbf{T}_\infty$.

(d) Let X be the set of all continuous real valued functions on the real line $\mathbf{R}$. Prove that X is metrizable such that the restriction maps from X to $\mathbf{BC}([-n, n])$ are uniformly continuous for all n. [*Hint:* Let $\mathbf{d}_n(f, g)$ be the maximum value of $|f(x) - g(x)|$ for $|x| \leq n$.]

(e) Given X and the metric constructed in the previous part of the problem, prove that a sequence of functions $\{ f_n \}$ converges to f if and only if for each compact subset $K \subset \mathbf{R}$ the sequence of restricted functions $\{ f_n|K \}$ converges to $f|K$.

(f) Is X complete with respect to the metric described above? Prove this or give a counterexample.

(g) Explain how the preceding can be generalized from continuous functions on $\mathbf{R}$ to continuous functions on an arbitrary open subset $U \subset \mathbf{R}^n$.

2. (a) Let U be an open subset of $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^n$ be a continuous function such that $f^{-1}(\{0\})$ is contained in an open subset V such that $V \subset \overline{V} \subset U$. Prove that there is a continuous function g from $S^n \cong (\mathbf{R}^n)^\bullet$ to itself such that $g|V = f|V$ and $f^{-1}(\{0\}) = g^{-1}(\{0\})$. [*Hint:* Note that

$$(\mathbf{R}^n)^\bullet \ - \ \{0\} \quad \cong \quad \mathbf{R}^n$$

and consider the continuous function on

$$\left(\overline{V} - V\right) \sqcup \{\infty\} \quad \subset \quad (\mathbf{R}^n)^\bullet \ - \ \{0\}$$

defined on the respective pieces by the restriction of f and ∞. Why can this be extended to a continuous function on $(\mathbf{R}^n)^\bullet \ - \ V$ with the same codomain? What happens if we try to piece this together with the original function f defined on U?]

(b) Suppose that we are given two continuous functions g and g' satisfying the conditions of the first part of this exercise. Prove that there is a continuous funtion

$$G : S^n \times [0,1] \longrightarrow S^n$$

such that $G(x,0) = g(x)$ for all $x \in S^n$ and $G(x,1) = g'(x)$ for all $x \in S^n$ (*i.e.*, the mappings g and g' are **homotopic**).

3. Let X be compact, and let $\mathcal{F}$ be a family of continuous real valued functions on X that is closed under multiplication and such that for each $x \in X$ there is a neighborhood U of X and a function $f \in \mathcal{F}$ that vanishes identically on U. Prove that $\mathcal{F}$ contains the zero function.

4.** Let X be a compact metric space, and let J be a nonempty subset of the ring $\mathbf{BC}(X)$ of (bounded) continuous functions on X such that J is closed under addition and subtraction, it is an *ideal* in the sense that $f \in J$ and $g \in \mathbf{BC}(X) \Longrightarrow f \cdot g \in J$, and for each $x \in X$ there is a function $f \in J$ such that $f(x) \neq 0$. Prove that $J = \mathbf{BC}(X)$. [*Hints:* This requires the existence of partitions of unity as established in Theorem 36.1 on pages 225–226 of Munkres; as noted there, the result works for arbitrary compact Hausdorff spaces, but we restrict to metric spaces because the course does not cover Urysohn's Lemma in that generality. Construct a finite open covering of X, say $\mathcal{U}$, such that for each $U_i \in \mathcal{U}$ there is a function $f_i \in J$ such that $f_i > 0$ on U_i. Let $\{\,\varphi_i\,\}$ be a partition of unity dominated by $\mathcal{U}$, and form $h = \sum_i \varphi_i \cdot f_i$. Note that $h \in J$ and $h > 0$ everywhere so that h has a reciprocal $k = 1/h$ in $\mathbf{BC}(X)$. Why does this imply that the constant function 1 lies in J, and why does the latter imply that everything lies in J?]

5.** In the notation of the preceding exercise, an ideal $\mathbf{M}$ in $\mathbf{BC}(X)$ is said to be a *maximal ideal* if it is a proper ideal and there are no ideals $\mathbf{A}$ such that $\mathbf{M}$ is properly contained in $\mathbf{A}$ and $\mathbf{A}$ is properly contained in in $\mathbf{BC}(X)$. Prove that there is a 1–1 correspondence between the maximal ideals of $\mathbf{BC}(X)$ and the points of X such that the ideal $\mathbf{M}_x$ corresponding to X is the set of all continuous functions $g : X \to \mathbf{R}$ such that $g(x) = 0$. [*Hint:* Use the preceding exercise.]

APPENDICES

Appendix A : Topological groups

(Munkres, Supplementary exercises following $ 22; *see also* course notes, Appendix D)

Munkres, § 26, pp. 170–172: 12, 13

Munkres, § 30, pp. 194–195: 18

Munkres, § 31, pp. 199–200: 8

Munkres, § 33, pp. 212–214: 10

Additional exercises

Let $\mathbf{F}$ be the real or complex numbers. Within the matrix group $\mathbf{GL}(n, \mathbf{F})$ there are certain subgroups of particular importance. One such subgroup is the *special linear group* $\mathbf{SL}(n, \mathbf{F})$ of all matrices of determinant 1.

0. Prove that the group $\mathbf{SL}(2, \mathbf{C})$ has no nontrivial proper normal subgroups except for the subgroup $\{\pm I\}$. [*Hint:* If N is a normal subgroup, show first thatif $A \in N$ then N contains all matrices that are similar to A. Therefore the proof reduces to considering normal subgroups containing a Jordan form matrix of one of the following two types:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \quad \begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$$

Here α is a complex number not equal to 0 or ± 1 and $\varepsilon = \pm 1$. The idea is to show that if N contains one of these Jordan forms then it contains all such forms, and this is done by computing sufficiently many matrix products. Trial and error is a good way to approach this aspect of the problem.]

Definition. The *orthogonal group* $\mathbf{O}(n)$ consists of all transformations in $\mathbf{GL}(n, \mathbf{R})$ that take each orthonormal basis for $\mathbf{R}^n$ to another orthonormal basis, or equivalently, the subgroup of all matrices whose columns form an orthonormal basis. It is an easy exercise in linear algebra to show that the determinant of all matrices in $\mathbf{O}(n)$ is ± 1. The *special orthogonal group* $\mathbf{SO}(n)$ is the subgroup of $\mathbf{O}(n)$ consisting of all matrices whose determinants are equal to $+1$. Replacing If we replace the real numbers $\mathbf{R}$ by the complex numbers $\mathbf{C}$ we get the unitary groups $\mathbf{U}(n)$ and the *special unitary groups* $\mathbf{SU}(n)$, which are the subgroups of $\mathbf{U}(n)$ given by matrices with determinant 1. The determinant of every matrix in $\mathbf{U}(n)$ is of absolute value 1 just as before, but in the complex case this means that the determinant is a complex number on the unit circle. In Appendix A the orthogonal and unitary groups were shown to be compact.

1. Show that $\mathbf{O}(1)$ is isomorphic to the cyclic group of order 2 and that $\mathbf{SO})(2)$ is isomorphic as a topological group to the circle group S^1. Conclude that $\mathbf{O}(2)$ is homeomorphic as a space to $\mathbf{SO}(2) \times \mathbf{O}(1)$, but that as a group these objects are not isomorphic to each other. [*Hint:* In Appendix D it is noted that every element of $\mathbf{O}(2)$ that is not in $\mathbf{SO}(2)$ has an orthonormal basis of eigenvectors corresponding to the eigenvalues ± 1. What does this say about the orders of such group elements?]

2. Show that $\mathbf{U}(1)$ is isomorphic to the circle group S^1 as a topological group.

3. For each positive integer n, show that $\mathbf{SO}(n)$, $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ are connected spaces and that $\mathbf{U}(n)$ is homeomorphic as a space (but not necessarily isomorphic as a topological group) to $\mathbf{SU}(n) \times S^1$. [*Hints:* In the complex case use the Spectral Theorem for normal matrices to show that every unitary matrix lies in the path component of the identity, which is a normal subgroup. In the real case use the results on normal form in Appendix D.]

4. Show that $\mathbf{O}(n)$ has two connected components both homeomorphic to $\mathbf{SO}(n)$ for every n.

5. Show that the inclusions of $\mathbf{O}(n)$ in $\mathbf{GL}(n, \mathbf{R})$ and $\mathbf{U}(n)$ in $\mathbf{GL}(n, \mathbf{C})$ determine 1–1 correspondences of path components. [*Hint:* Show that the Gram-Schmidt orthonormalization process expresses an arbitrary invertible matrix over the real numbers as a product PQ where P is upper triangular and Q is orthogonal (real case) or unitary (complex case), and use this to show that every invertible matrix can be connected to an orthogonal or unitary matrix by a continuous curve.]