I. Foundational Material

Since this is a graduate course in mathematics, it is appropriate to assume some background material as well as some experience in working with it. In particular, we shall presume that the reader has a working knowledge of basic set theory; however, familiarity with a fully developed axiomatic approach to that subject will not be needed. For the sake of completeness, the following files on foundational material have been placed into the course directory:

`foundations1.*` (where $* =$ `ps` or `pdf`) **:** This is a review of mathematical logic and simple set-theoretic algebra.

`foundations2.*` (where $* =$ `ps` or `pdf`) **:** This discusses some material in set theory that either deviates slightly from the treatment in Munkres or is not covered there. Also included are discussions of the Axiom of Choice and the Generalized Continuum Hypothesis.

`realnumbers.*` (where $* =$ `ps` or `pdf`) **:** This gives a list of axioms for the real numbers and some important properties, including those that are play a major role in this course.

`uniqreals.*` (where $* =$ `ps` or `pdf`) **:** This proves that the axioms for the real numbers completely characterize the latter up to a structure preserving one-to-one correspondence (*i.e*,, an **isomorphism**). One needs to prove a result of this sort in order to talk about **THE** real number system. The uniqueness proof is not particularly difficult but it is rather tedious; given the importance of uniqueness, it is worthwhile to look through this argument sometime and to understand it at least passively.

`categories.*` (where $* =$ `ps` or `pdf`) **:** Category theory will not be used explicitly in this course, but given its fundamental nature a discussion of the basic ideas and some examples is included for reference.

Most of the discussion of foundational material here will concentrate on points that are not worked out in detail in Munkres but are important for the course.

I.1 : Basic set theory

(Munkres, §§ 1, 2, 3)

As indicated on page 3 of Munkres, our approach to questions in logic and set theory is to assume some familiarity with the most elementary ideas and to discuss what else is needed without spending so much time on these subjects that important material in topology must be omitted.

Munkres mentions that an overly casual approach to set theory can lead to logical paradoxes. For example, this happens if we try to consider the "set of all sets." During the early part of the twentieth century mathematicians realized that problems with such things could be avoided by stipulating that sets cannot be "too large," and effective safeguards to eliminate such difficulties were built into the formal axioms for set theory. One simple and reliable way of avoiding such problems with the informal approach to set theory in this course this is to assume that all constructions take place in some extremely large set that is viewed as universal. This is consistent with the formal axiomatic approach, where one handles the problem by considering two types of collections of objects: The **CLASSES** can be fairly arbitrary, but the **SETS** are constrained by a simple logical condition (specifically, they need to belong to *some* other class). Viewing everything as contained in some very large set is the recommended option for this course if difficulties ever arise.

I.2 : Products, relations and functions

(Munkres, §§ 5, 6, 8)

The main differences between this course and Munkres are outlined in the first few pages of `foundations2.*` .

Definitions relations and functions

Since functions play such an important role in topology, it is appropriate to stress the differences between the definition of functions in this course and the definitions that appear in many mathematical textbooks.

Our definition of a function is equivalent to that of Munkres, but the crucial difference between the definitions of relations and functions in Munkres and this course is that the source set (or *domain*) and the target set (or *codomain*) are explicit pieces of data in the definition of a function. Generally the source or domain is unambiguous because usually (but not always!) it is the set on which the function is defined. However, the target or codomain is often ignored. The target must contain the image of the function, but for each set containing the image there is a different function whose target is the given set. For example, this means that we distinguish between the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to [0, 1]$ defined by the same formula:

$$f(x) = g(x) = \frac{1}{x^2 + 1}$$

One particular context in which it is necessary to make such distinctions is the construction of the fundamental group as in Chapter 9 of Munkres.

Comments on terminology

Mathematicians have two alternate lists of words for discussing functions that are 1–1 and onto, and for set theory and many other branches of mathematics these are used interchangeably. One of these lists is given on page 18 of Munkres: A map if *injective* if it is 1–1, *surjective* if it is onto, and *bijective* if it is both (*i.e.*, a 1–1 correspondence). The terms *monomorphism* and *epimorphism* from category theory are often also used in set theory as synonyms for 1–1 and onto. However, some care is needed when using these category-theoretic terms for functions in discussing morphisms of topological objects, so we shall (try to) avoid using such terminology here.

Disjoint unions

We shall also use an elementary set-theoretic construction that does not appear in Munkres; namely, given two sets A and B we need to have a *disjoint union*, written $A \sqcup B$ or $A \coprod B$, which is a union of two disjoint subsets that. are essentially xerox copies of A and B. One use of this construction will appear in the discussion of cardinal numbers; other applications will be discussed in the main body of the course (in particular, see the section, "Sums and cutting and pasting").

I.3 : Cardinal numbers

(Munkres, §§ 4, 7, 9)

The theory of cardinal numbers for infinite sets illustrates that set theory is not just a more sophisticated approach to well-known mathematical concepts and that it is also a powerful tool for obtaining important new insights into mathematics.

Definition. Two sets A and B have the same cardinality (or cardinal number) if there is a 1–1 onto map (or 1–1 correspondence) $f : A \to B$. Frequently we write $|A| = |B|$ in this case; it follows immediately that this relation is reflexive, symmetric and transitive.

We begin with two basic properties of infinite sets:

PROPOSITION. *Every infinite set contains a subset that is in $1-1$ correspondence with the positive integers.*∎

GALILEO'S PARADOX. *A set A is infinite if and only if there is a $1-1$ correspondence between A and a proper subset of A.*∎

Partial ordering of cardinalities

Definition. If A and B are sets, we write $|A| \leq |B|$ if there is a 1–1 map from A to B.

It follows immediately that this relation is transitive and reflexive, but the proof that it is symmetric is decidedly nontrivial:

SCHRÖDER-BERNSTEIN THEOREM. *If A and B are sets such that there are $1-1$ maps $A \to B$ and $B \to A$, then $|A| = |B|$.*

Sketch of proof. This is the classic argument from Birkhoff and MacLane's *Survey of Modern Algebra* (see page 340 in the Third Edition).

Let $f : A \to B$ and $g : B \to A$ be the 1–1 mappings. Each $a \in A$ is the image of at most one *parent* element $b \in B$; in turn, the latter (if it exists) has at most one parent element in A, and so on. The idea is to trace back the ancestry of each element as far as possible. For each point in A or B there are exactly three possibilities; namely, the ancestral chain may go back forever, it may end in A, or it may end in B.

Split A and B into three pieces corresponding to these cases, and call the pieces A_1, A_2, A_3 and B_1, B_2, B_3 (the possibilities are ordered as in the previous paragraph).

The map f defines a 1–1 correspondence between A_1 and B_1 (and likewise for g). Furthermore, g defines a 1–1 correspondence from B_2 to A_2, and f defines a 1–1 correspondence from B_3 to A_3. If we combine these 1–1 correspondences $A_1 \longleftrightarrow B_1$, $A_2 \longleftrightarrow B_2$ and $A_3 \longleftrightarrow B_3$, we obtain a 1–1 correspondence between all of A and all of B.∎

If we are only considering subsets of some large set, then we can define the "cardinal number" of a subset to the the equivalence class of that set under the relation $A \sim B \leftrightarrow |A| = |B|$. If one assumes the Axiom of Choice it is possible to give an equivalent definition of cardinal number that extends to all sets, but for the time being we shall not worry about this point.

The Schröder-Bernstein Theorem and the preceding observations imply that *if A is an infinite subset of the positive integers $\mathbf{N}^+$, then* $|A| = |\mathbf{N}^+|$. Since we also know that $|A| \le |\mathbf{N}^+|$ for every infinite set A, it follows that $|\mathbf{N}^+|$ can be viewed as the unique smallest infinite cardinal number and that it is $\le$ every other infinite cardinal number. Following Cantor's notation this cardinal number is generally denoted by $\aleph_0$ (aleph-null).

In contrast, Cantor's Diagonal Process technique implies that in effect there is no largest infinite cardinal number.

THEOREM. *If A is a set and $\mathbf{P}(A)$ is the set of all subsets of A then* $|A| < |\mathbf{P}(A)|$.

Sketch of proof. The first step is to notice that $\mathbf{P}(A)$ is in 1–1 correspondence with the set of all set-theoretic functions $A \to \{0, 1\}$. Given a subset B let χ_B be the *characteristic function of* B that is 1 on B and 0 on $A - B$. Conversely given a function j of the type described, if we set $B = j^{-1}(\{1\})$ then $j = \chi_B$.

The map $A \to \mathbf{P}(A)$ sending a to $\{a\}$ is a 1–1 mapping, and therefore $|A| \le |\mathbf{P}(A)|$. Therefore it is only necessary to prove that the cardinalities are unequal. The Diagonal Process argument is worked out in the proof of Theorem 7.7 on pages 49–50 of Munkres in the special case where $A = \mathbf{N}^+$, and the same argument works for an arbitrary nonempty set.■

Notational conventions. If one assumes the Axiom of Choice then one can show that the class of all cardinal numbers is linearly ordered, and in fact it is well-ordered (every nonempty subcollection has a least element). Of course, the minimum element of the class of infinite cardinals is $\aleph_0$, and one proceeds similarly to define $\aleph_1$ to be the least infinite cardinal in the complement of $\{\aleph_0\}$, $\aleph_2$ to be the least infinite cardinal in the complement of $\{\aleph_0, \aleph_1\}$, and similarly one can define $\aleph_n$ recursively for each positive integer n.

We now claim that there is a set S whose cardinality is greater than $\aleph_n$ for all n. Choose A_n such that $|A_n| = \aleph_n$ and consider $S = \cup_n A_n$. By construction we must have $|S| > |A_k|$ for all k. Therefore one can then define $\aleph_\omega$ to be the first infinite cardinal not in $\{\aleph_0, \aleph_1, \aleph_2, \cdots\}$. Similar considerations show that for any **SET** of cardinal numbers one can always find a larger one.

Cardinal arithmetic

One can perform a limited number of arithmetic operations with cardinal numbers, but it is necessary to realize that these do not enjoy all the familiar properties of the corresponding operations on positive integers.

Definition. If A and B are sets, then

(*i*) the **sum** $|A| + |B|$ is equal to $|A \sqcup B|$,

(*ii*) the **product** $|A| \cdot |B|$ is equal to $|A \times B|$,

(*iii*) the **exponentiation** $|A|^{|B|}$ is equal to $|\mathbf{F}(B, A)|$, the set of all set-theoretic functions from B to A.

If A and B are finite, the first two are true by simple counting arguments, and the third is true because the set of functions is in 1–1 correspondence with a product of $|B|$ copies of A. Note that the Diagonal Process argument above shows that $|\mathbf{P}(A)| = 2^{|A|}$.

The following simple result illustrates the difference between finite and infinite cardinals:

PROPOSITION. *If A is finite and B is infinite, then* $|A| + |B| = |B|$.

Sketch of proof. Let $C \subset B$ be a subset in 1–1 correspondence with $\mathbf{N}^+$. Clearly

$$|A \sqcup C| = |A \sqcup \mathbf{N}^+| = |\mathbf{N}^+| = |C|$$

and one can use this to construct a 1–1 correspondence between B and $A \sqcup B$.■

The following standard identities involving $\aleph_0$ were first noted by Galileo and Cantor respectively.

THEOREM. *We have* $\aleph_0 + \aleph_0 = \aleph_0$ *and* $\aleph_0 \cdot \aleph_0 = \aleph_0$.

The first of these is easily seen by splitting $\mathbf{N}^+$ into the even and odd positive integers, while the traditional diagonal argument for proving the latter is worked out in Exercise 2 on page 45 of Munkres.■

We now have the following standard consequences.

COROLLARY. *If* $\mathbf{Z}$ *and* $\mathbf{Q}$ *are the integers and rationals respectively then* $|\mathbf{Z}| = |\mathbf{Q}| = \aleph_0$.■

These lead directly to the following fundamental result.

THEOREM.

$$|\mathbf{R}| = \left|\mathbf{P}\left(\mathbf{N}^+\right)\right| = 2^{\aleph_0}$$

Sketch of proof. Note that Munkres does not prove this result explicitly although a proof is indicated in the exercises and on page 177 the real numbers are shown to have cardinality greater than $\aleph_0$.

Usually this is derived using decimal expansions of real numbers, but we shall give a proof that does not involve decimals (although the idea is similar). The idea is to construct 1–1 maps from $\mathbf{R}$ to $\mathbf{P}\left(\mathbf{N}^+\right)$ and vice versa and then to apply the Schröder-Bernstein Theorem.

Let $D : \mathbf{R} \to \mathbf{P}(\mathbf{Q})$ be the Dedekind cut map sending a real number r to the set of all rational numbers less than r. Since there is always a rational number between any two distinct real numbers, it follows that this map is 1–1. Since $|\mathbf{Q}| = \aleph_0$ there is a 1–1 correspondence from $\mathbf{P}(\mathbf{Q})$ to $\mathbf{P}\left(\mathbf{N}^+\right)$, and the composite of D with this map gives the desired 1–1 map from $\mathbf{R}$ to $\mathbf{P}\left(\mathbf{N}^+\right)$.

Let $\mathbf{P}_\infty\left(\mathbf{N}^+\right)$ denote the set of all *infinite* subsets of $\mathbf{N}^+$, and define a function from $\mathbf{P}_\infty\left(\mathbf{N}^+\right)$ to $\mathbf{R}$ as follows: Given an infinite subset B let χ_B be its characteristic function and consider the infinite series

$$\sum_B = \sum_k \chi_B(k) \cdot 2^{-k} \ .$$

This series always converges because its terms are nonnegative and less than 1 (and the series $\sum_k 2^{-k}$ converges), and different *infinite* subsets yield different values (look at the first value of k that is in one subset but not in the other; if, say, k lies in A but not in B then we have $\sum_A > \sum_B$). Note that $\sum_A \in [0,1]$ for all infinite subsets A since $\sum_k 2^{-k} = 1$.

If A is a *finite* subset, consider the finite sum

$$\Theta_B = 2 + \sum_k \chi_B(k) \cdot 2^{-k} \ .$$

Once again it follows that different finite subsets determine different real (in fact, rational) numbers. Furthermore, since the value associated to a finite set lies in the interval $[2,3]$ it is clear that a finite set and an infinite set cannot go to the same real number. Therefore we have constructed a 1–1 function from $\mathbf{P}(\mathbf{N}^+)$ to $\mathbf{R}$.

Since we have constructed 1–1 mappings both ways, we can use the Schröder-Bernstein Theorem to complete the proof.■

Natural question. The Continuum Hypothesis states that $|\mathbf{R}| = \aleph_1$; important results of P. Cohen show that one can construct models of set theory for which this statement is true and other models for which it is false. One an ask which cardinal numbers are possible values for $|\mathbf{R}|$. Results on this and more general questions of the same type can also be settled using the methods introduced by Cohen. In particular, it turns out that $|\mathbf{R}|$ can be equal to $\aleph_n$ for every positive integer n but it cannot be equal to $\aleph_\omega$ (all these are defined above). A proof of the last assertion appears in the exercises on page 66 of the book, *Set Theory and Metric Spaces*, by I. Kaplansky.

Finally, we prove the another fundamental and well known result about the cardinality of $\mathbf{R}$:

PROPOSITION. *For all positive integers n we have $|\mathbf{R}^n| = |\mathbf{R}|$.*

Using the Axiom of Choice one can show that $|A \sqcup A| = |A|$ and $|A^n| = |A|$ for every infinite set A and positive integer n, but we shall outline a direct and relatively standard argument.

Sketch of proof. There is a generalization of a familiar law of exponents for cardinal numbers

$$\gamma^{\alpha+\beta} = \gamma^\alpha \cdot \gamma^\beta$$

that follows from the 1–1 correspondence

$$\mathbf{F}(A \sqcup B, C) \to \mathbf{F}(A, C) \times \mathbf{F}(B, C)$$

which sends a function f to

$$(f \circ i_A, f \circ i_B)$$

(recall that i_A and i_B are the injections from A and B to $A \sqcup B$).

An inductive argument shows that it suffices to prove the result when $n = 2$. In this case the proof becomes a completely formal exercise involving the exponential law described above:

$$|\mathbf{R}^2| = |\mathbf{R}| \times |\mathbf{R}| = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0}$$

COROLLARY. *We also have $2^{\aleph_0} + 2^{\aleph_0} = 2^{\aleph_0}$ and $\aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0}$.*

Proof. These are consequences of the following chain of inequalities:

$$2^{\aleph_0} \quad \leq \quad 2^{\aleph_0} + 2^{\aleph_0} \quad \leq \aleph_0 \cdot 2^{\aleph_0} \quad \leq$$

$$2^{\aleph_0} \cdot 2^{\aleph_0} \quad = \quad 2^{\aleph_0} \ ■$$

Remarks. 1. The following generalizations of the usual laws of exponents also hold for cardinal numbers:

$$\gamma^{\alpha\beta} = \left(\gamma^\alpha\right)^\beta, \qquad \left(\beta \cdot \gamma\right)^\alpha = \beta^\alpha \cdot \gamma^\alpha$$

Since the proofs require some lengthy (but relatively elementary) digressions and we shall not need these results in this course, we shall not verify these relationships.

2. Another natural question about cardinal arithmetic is whether $2^\alpha = 2^\beta$ implies $\alpha = \beta$ as is the case for nonnegative integers. If the Generalized Continuum Hypothesis is true, then the answer is yes. On the other hand, this condition is not strong enough to imply the Generalized Continuum Hypothesis, and one can also construct models of set theory for which $\alpha < \beta$ but $2^\alpha = 2^\beta$. More generally, very strong results on the possible sequences of cardinal numbers that can be written as 2^α for some α are given by results of W. B. Easton that build upon P. Cohen's work on the Continuum Hypothesis; the result essentially states that are few straightforward necessary conditions on such sequences are also sufficient. These results first appeared in in the following paper by Easton: *Powers of regular cardinals*, Ann. Math Logic **1** (1970), 139–178. A more recent paper by T. Jech covers subsequent work on this problem: *Singular cardinals and the PCF theory*, Bull. Symbolic Logic **1** (1995), 408–424.

I.4 : The real number system

(Munkres, § 4)

The first Chapter of Rudin, *Principles of Mathematical Analysis* (Third Edition), contains detailed arguments for many of the results about real numbers from the files in the course directory. Aside from the usual identities involving addition, subtraction, multiplication, division and inequalities, the most crucial properties of the real numbers throughout the course are the following:

(1) **Completeness.** Every nonempty subset of $\mathbf{R}$ that has an upper bound has a **least** upper bound, and every nonempty subset of $\mathbf{R}$ that has a lower bound has a **greatest** upper bound.

(2) **Density of rational numbers.** If we are given two real numbers a and b such that $a < b$, then there is a rational number q such that $a < q < b$.

(3) If $\varepsilon > 0$ then there is a positive integer n such that $\frac{1}{n} < \varepsilon$.