II. Metric and topological spaces

The discussion of the main topics in this course begins here.

As noted in a classic text on general topology by W. Franz, " the word 'topology' is derived from the Greek word $\tau \acute{o} \pi o \varsigma$ which means 'place,' 'position,' or 'space' ... it is a subdiscipline of geometry" (see pp.1–3 of the book by Franz; complete bibliographic information is given at the end of these notes).

Although a detailed discussion of the history of point set topology is beyond the scope of these notes, it is useful to mention two important points that motivated the original development of the subject.

(1) Several important theorems about continuous functions of real valued functions of a (single) real variable have analogs in other contexts, and the most effective way to work with such analogs is to develop a unified approach.

(2) When one considers functions of several real variables, more thought must be given to the sets on which functions are defined. For one variable, the emphasis is on functions defined over some interval, which may or may not have end points. On the other hand, for functions of two or more real variables there is an overwhelming variety of shapes to consider (*e.g.*, round, square, triangular, hexagonal, octagonal, with and without some or all boundary points, with or without holes inside, U-shaped, X-shaped, Y-shaped, ...). Clearly there are far too many to enumerate in a relatively simple manner. Therefore, when one wants to discuss concepts like partial differentiation, it is better to begin by considering a reasonable class of **regions** or **domains** to work with. Mathematically, these conditions are given by the definition of an **open set**.

The concepts of point set topology have proven to be useful — in fact indispensable — in a wide range of mathematical contexts where it is meaningful to talk about two objects being close to each other in some algebraic, analytic or geometric sense. In particular, these concepts have played a major and foundational role in the application of geometric ideas to solve analytical questions, the interactions between the two subjects have stimulated each one to a great extent (interactions with algebra have also been mutually beneficial). A graduate course in point set topology should take the important links with algebra and analysis into account, but it also seems important to retain as much of the geometric nature of the subject as possible, particularly for a course that is the first third of a full year sequence, and striking a decent balance from a contemporary perspective is one goal of these notes.

For the sake of completeness, here are some references on the history of topology and related topics:

`www.math.uiuc.edu/~droyster/courses/fall99/math4181/classnotes/notes1.pdf`

`www-gap.dcs.st-and.ac.uk/~history/Hist/Topics/Topology_in_mathematics.html`

`www.wikipedia.org/wiki/Topology`

History of Topology, I. M. James (ed.)

Full information on this and other books cited in these notes appears in the bibliography at the end.

II.1 : Metrics and topologies

(Munkres, §§ 12, 13, 16, 20; Edwards, § I.7)

For both historical and logical reasons one can view the most basic aspects of point set theory as natural generalizations of important properties of certain subsets of the real numbers. One approach to doing this is to base the discussion on an abstract notion of **DISTANCE** that generalizes the usual notion of distance between two numbers in the obvious fashion. Another approach is to take the concept of an **OPEN SUBSET** (or something logically equivalent) as the fundamental abstract structure. It is not difficult to formulate the concept of an open set if one has a notion of distance, so there is a natural progression of abstraction from the real numbers to *metric spaces* (sets with a suitable notion of distance) to *topological spaces* (sets with a suitable notion of open subsets).

Eventually a course in point set topology needs to cover both types of structures, but there is no universal agreement on which should come first and when the other should be introduced. The approach in these notes will be to introduce metric spaces first and topological spaces immediately afterwards. This will allow us to take advantage of the strengths of both approaches throughout the course.

The basic definitions and a few examples

The notion of distance between two points in $\mathbf{R}^n$ is fundamentally important in multivariable calculus and some aspects of linear algebra. It turns out that an extremely short list of properties for distances are enough to prove abstract versions of many important results from advanced calculus and real variables courses.

Definition. A *metric space* is a pair $(X, \mathbf{d})$ consisting of a set X and a function $\mathbf{d} : X \times X \to \mathbf{R}$ (sometimes called the **metric** or **distance function**, with $\mathbf{d}(x, y)$ being called the **distance** from x to y, or between x and y) such that the following properties hold:

(MS1) $\mathbf{d}(x, y) \geq 0$ for all $x, y \in X$.
(MS2) $\mathbf{d}(x, y) = 0$ if and only if $x = y$.
(MS3) $\mathbf{d}(x, y) = \mathbf{d}(y, x) \geq 0$ for all $x, y \in X$.
(MS4) $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$ for all $x, y, z \in X$.

The last property is often called the *triangle inequality* because it generalizes the usual triangle inequality from classical Euclidean geometry.

EXAMPLES. 1. The most important examples are the ordinary coordinate or Euclidean spaces $\mathbf{R}^n$ for which $\mathbf{d}(x, y) = |x - y|$. The four basic properties for an abstract metric are established in undergraduate courses containing linear or vector algebra.

2. If $(X, \mathbf{d})$ is a metric space and A is a subset of X, then one can make A into a metric space using the *subspace metric* given by $\mathbf{d}\big|(A \times A)$; less formally, this means that the distances between points of A are the same as their distances in X itself.

3. If S is an arbitrary set, then one can make S into a metric space with the so-called *discrete metric*, for which $\mathbf{d}(s, t) = 1$ if $s \neq t$ and 0 if $s = t$. It is a routine exercise to verify that this defines a metric on x.

4. One can for an abstract generalization of the first example a follows: If one defines a *norm* on a real vector space V to be a function sending each $v \in V$ to a nonnegative real number $|v|$ such that

(a) $|v| = 0$ if and only if $v = 0$,
(b) $|cv| = |c||v|$ for all $c \in \mathbf{R}$ and $v \in V$,
(c) $|v + w| \leq |v| + |w|$ for all $v, w \in V$,

then the formula $\mathbf{d}(v, w) = |v - w|$ defines a metric on V. It is again an elementary exercise to verify that this satisfies the four conditions required for a metric.

5. We need to give some additional examples in order to show that the preceding construction yields something beyond ordinary Euclidean spaces.

(5A) Actually, this is two examples. Take $V = \mathbf{R}^n$, write a typical vector x in coordinates as $(x_1, \cdots, x_n)$, and consider the functions $|x|_1 = \sum_i |x_i|$ and $|x|_\infty = \max_i\{\ |x_i|\ \}$. It is again elementary to check that each of these define norms. If one wants to distinguish the previous norm from these it is customary to write the latter as $|x|_2$, which reflects the quadratic nature of the latter (as in measure theory, one can interpolate an entire series of norms $|x|_p$ for $1 \leq p \leq \infty$ but we shall not need these examples here).

(5B) This is a much larger example. Let S be a set, and let $\mathbf{C}(S)$ be the space of all bounded real valued functions on S. Then a norm is defined by the formula $|f| = \sup_{x \in S}\{\ |f(x)|\ \}$.

It is also possible to construct a vast array of other norms on vector spaces at this point, but we shall not do so in order to avoid straying too far from the central themes of the course.

The examples above show that one can construct many different metrics on a given set. However, it clearly becomes very cumbersome to write $(X, \mathbf{d})$ every time we are referring to a metric space, so in order to simplify the exposition we shall often simply write X if the metric is clear from the context.

Open sets

The basic definition extends the one for Euclidean spaces.

Definition. Let $(X, \mathbf{d})$ be a metric space. A subset $U \subset X$ is said to be *open* if for each $x \in U$ there is a positive real number ε such that $\mathbf{d}(x, y) < \varepsilon \Longrightarrow y \in U$.

For each $r > 0$ and $x \in X$, the set

$$N_r(y) = \{\ y \in X \mid \mathbf{d}(x, y) < r\ \}$$

is called the open ball (or disk or neighborhood) of radius r centered at x. One can rewrite the definition of open set to say that for all $x \in U$ there exists an $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$.

The most important properties of open sets in metric spaces are summarized in the following result:

THEOREM. *Let $\mathcal{U}$ be the family of all open subsets of X. The the following hold:*

(i) *Each subset of the form $N_\varepsilon(x)$ is open in X.*
(ii) *The empty set and X itself are both open in X.*
(iii) *If for each $\alpha \in A$ the set U_α is open in X then $\cup_\alpha U_\alpha$ is also open in X.*
(iv) *If U_1 and U_2 are open in X then $U_1 \cap U_2$ is also open in X.*

One can combine (iv) and finite induction to prove that the intersection of any finite collection of open subsets of X is also open in X.

Proof. (i) Let $y \in N_\varepsilon(x)$, and let $s = \mathbf{d}(x,y)$. We claim that $N_{\varepsilon-s}(y) \subset N_\varepsilon(x)$. It may be worthwhile to draw a two-dimensional picture at this point in order to make this assertion plausible; the formal proof of the assertion proceeds as follows. Suppose that $z \in N_{\varepsilon-s}(y)$, so that $\mathbf{d}(y,z) < \varepsilon - s$. We need to prove that $\mathbf{d}(x,z) < \varepsilon$. Applying the triangle inequality we have

$$\mathbf{d}(x,z) \leq \mathbf{d}(x,y) + \mathbf{d}(y,z) = s + \mathbf{d}(y,z) < s + (\varepsilon - s) < \varepsilon$$

as required.

(ii) We shall first consider the case of the empty set. It has no points so the condition on all points in it will automatically be true because it is a statement about nothing. The openness of x follows because $N_\varepsilon(x) = X$ for all x and ε.

(iii) Suppose that $x \in \cup_\alpha U_\alpha$, and choose $\beta \in A$ so that $x \in U_\beta$. Then one can find $\varepsilon > 0$ so that $N_\varepsilon(x) \subset U_\beta$, and since $U_\beta \subset \cup_\alpha U_\alpha$ it also follows that $N_\varepsilon(x) \subset \cup_\alpha U_\alpha$.

(iv) Let $i = 1$ or 2, and let $x \in U_1 \cap U_2$. Then one has $\varepsilon_i > 0$ so that $N_{\varepsilon_i}(x) \subset U_i$ for $i = 1, 2$. If we take ε to be the smaller of ε_1 and ε_2 then $N_\varepsilon(x) \subset U_1 \cap U_2$. Once again, it might be helpful to draw a picture as an aid to understanding this proof.■

It is useful to look at the meaning of open subset for one of the examples described above; namely, the discrete metric on a set. In this case $N_1(x)$ is merely the one point set $\{x\}$. Thus every one point subset of S is open with respect to the discrete metric. But if $W \subset S$, then clearly we have

$$W = \bigcup_{x \in W} \{x\}$$

so by the preceding theorem we see that *every subset of a metric space with a discrete metric is an open subset.* Of course, for examples like Euclidean spaces there are many examples of subsets that are not open. In particular, one point subsets are **NEVER** open in Euclidean spaces (unless one adopts the convention $\mathbf{R}^0 = \{0\}$, in which case this object must be excluded).

Topological spaces

The preceding theorem provides the motivation for the central concept of a course in point set topology:

Definition. A *topological space* is a pair $(X, \mathbf{T})$ consisting of a set X and a collection $\mathbf{T}$ of subsets of X satisfying the following conditions:

(TS1) The empty set and X itself both belong to $\mathbf{T}$.
(TS2) If for each $\alpha \in A$ the set $U_\alpha \subset X$ belongs to $\mathbf{T}$, then $\cup_\alpha U_\alpha$ also belongs to $\mathbf{T}$.
(TS3) If U_1 and U_2 belong to $\mathbf{T}$, then $U_1 \cap U_2$ also belongs to $\mathbf{T}$.

We often say that $\mathbf{T}$ is a topology on X or that $\mathbf{T}$ is the family of open subsets of X, and we say that $U \subset X$ is open if $U \in \mathbf{T}$. As before, if the topology on a set is clear from the context we shall often use the set by itself to denote a topological space.

Note. One can combine **(TS3)** and finite induction to prove that the intersection of any finite collection of subsets in $\mathbf{T}$ is also open in $\mathbf{T}$.

If $(X, \mathbf{d})$ is a metric space and $\mathbf{T}$ denotes the family of open subsets of X, then by the preceding theorem $(X, \mathbf{T})$ is automatically a topological space. We often call this **the metric topology** (associated to $\mathbf{d}$).

In particular, if S is an arbitrary set and we put the discrete metric on S, then we have seen that all subsets of S are open in the corresponding metric topology. More generally, a topological space is said to be *discrete* if the every subset is open (equivalently, every one point subset is open); by previous observations, this is just the metric topology associated to the discrete metric.

On the other hand, there are many examples of topological spaces that do not come from metric spaces.

EXAMPLES. 1. Given a set X, the *indiscrete topology* on X is the family $\mathbf{T}$ consisting only of the empty set and X itself. It is elementary to verify that this defines a topology on X. However, if X contains at least two points then this cannot come from a metric space because if X is a metric space and $p \in X$ then $X - \{p\}$ is open for all $p \in X$ (we shall prove this below).

2. Given a set X, the *finitary topology* on X is the family $\mathbf{T}$ consisting of the empty set and all subsets of the form $X - A$ where A is finite. The verification that $\mathbf{T}$ is a topology can be found in Example 3 on page 77 of Munkres. If X is finite this is equal to the metric topology for the discrete metric. On the other hand, if X is infinite, then if u and v are distinct points of X and U and V, then $U \cap V$ is always infinite (its complement is finite!), and by the Hausdorff separation property below it follows that $\mathbf{T}$ does not come from a metric on X if the latter is infinite.

Here are the results that we need to show that these examples do not come from metrics:

PROPOSITION T1. *If X is a metric space and $p \in X$, then $X - \{p\}$ is open.*

Proof. Let $q \in X - \{p\}$, so that $q \neq p$ and $r = \mathbf{d}(q,p) > 0$. Then clearly $N_r(q) \subset X - \{p\}$, and hence the latter is open.

PROPOSITION T2. (Hausdorff Separation Property) *If X is a metric space and $u, v \in X$ are distinct points, then there exist disjoint open subsets U and V containing u and v respectively.*

Proof. Let $2\varepsilon = \mathbf{d}(x,y) > 0$, and take U and V to be $N_\varepsilon(u)$ and $N_\varepsilon(v)$ respectively. To see that these are disjoint, suppose that they do have some point z in common. Then by the triangle inequality and $z \in N_\varepsilon(u) \cap N_\varepsilon(v)$ we have

$$2\varepsilon = \mathbf{d}(u,v) \leq \mathbf{d}(u,z) + \mathbf{d}(z,v) < \varepsilon + \varepsilon$$

which is a contradiction. Therefore the intersection must be empty. Once again, it might be helpful to draw a picture as an aid to understanding this proof.

The following result is often useful in working with open sets:

LEMMA. *If U is an open subset of a metric space, then one can find numbers $\varepsilon(y) > 0$ for all $y \in U$ such that $U = \cup_y \; N_{\varepsilon(y)}(y)$.*

Proof. If $\varepsilon(y) > 0$ such that $N_{\varepsilon(y)}(y) \subset U$, then we have the chain of inclusions

$$U = \bigcup_y \{y\} \;\subset\; \bigcup_y \; N_{\varepsilon(y)}(y) \;\subset\; U$$

shows that $U = \cup_y \; N_{\varepsilon(y)}(y)$.

Comparing and constructing topologies on a set

We have seen that a given set may have several different metrics and several different topologies. There are a few aspects of the topologies on a space that are worth examining in some more detail at this time.

Since topologies on a set are just families of subspaces, it is meaningful to ask if one is contained in the other. Every topology must contain the indiscrete topology, and since the topology of a discrete metric contains every subset it is clear that every other topology is contained in this one. Frequently one sees statements that one topology $\mathbf{T}$ on a space is stronger or weaker than another topology $\mathbf{S}$ if one contains the other, and the terms coarser or finer are also used in such contexts. Unfortunately, this notation can be hopelessly confusing because, say, there is no consistency about whether a stronger topology has more open sets than a weaker one or vice versa. We shall avoid this by simply saying that one topology is larger or stronger than the other (compare the first few lines on page 78 of Munkres).

The following observation is elementary to verify:

FACT. *The unions and intersections of arbitrary families of topologies on a given set X are also topologies on X.*∎

Another abstract feature of topologies is that *given a family $\mathcal{A}$ of subsets of X, there is a unique minimal topology $\mathbf{T}(\mathcal{A})$ on X that contains $\mathcal{A}$.*

In view of the fact stated above, the topology can be described as the intersection of all topologies that contain $\mathcal{A}$; this family is nonempty because the discrete topology is a specific example of a topology containing $\mathcal{A}$. However, for many purposes it is necessary to have a more explicit description of this topology. Define $\mathcal{A}^*$ to be the set of arbitrary unions of sets having the form $A_1 \cap \cdots \cap A_k$ for some finite family of subsets $\{A_1, \cdots, A_k\}$ in $\mathcal{A}$ together with X and the empty set. To show that $\mathcal{A}^*$ is a topology it is necessary to verify that it is closed under arbitrary unions and finite intersections. It will be convenient to let $\mathcal{B}$ denote the set of finite intersections of sets in $\mathcal{A}$; then every element of $\mathcal{A}^*$ except possibly X and the empty set will be a union of subsets in $\mathcal{B}$. Since a union of unions of subsets in $\mathcal{B}$ is again a union of subsets in $\mathcal{B}$, it follows that $\mathcal{A}^*$ is closed under taking arbitrary unions. Suppose now that U and V lie in $\mathcal{A}^*$. Write these sets as $\cup_\beta U_\beta$ and $\cup_\gamma V_\gamma$ respectively; then

$$U \cap V = \bigcup_{\beta,\gamma} U_\beta \cap V_\gamma$$

where each summand $U_\beta \cap V_\gamma$ is a finite intersection of subsets in $\mathcal{A}$, and this implies that $U \cap V \in \mathcal{A}^*$ as required.

Definition. A family $\mathcal{A}$ of subsets of X is called a *subbase* for the topology $\mathbf{T}$ on X if $\mathcal{A}^* = \mathbf{T}$.

Basic open subsets for a topology

There is a special type of subbase known as a *base* that is often useful in constructing topologies.

Definition. A family $\mathcal{B}$ of subsets of X is called a *base* for the topology $\mathbf{T}$ on X if

(B0) $\mathcal{B}^* = \mathbf{T}$,
(B1) each $x \in X$ belongs to at least one $B \in \mathcal{B}$,

(B2) if $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ is a base for $\mathbf{T}$ we often refer to the sets in $\mathcal{B}$ as *basic open subsets.*

The following result is elementary to prove:

PROPOSITION. *If $\mathcal{B}$ is a family of subsets of X satisfying* **(B1)** *and* **(B2)** *above, then the smallest topology containing $\mathcal{B}$ is the set of all unions of sets in $\mathcal{B}$ together with the empty set.*■

Lemma 13.2 on page 80 of Munkres is another useful result on bases for topologies.

Subspace topologies

We have already noted that a subset of a metric space can be viewed in a natural way as a metric space in its own right by restricting the metric. There is a parallel way of viewing a subset of a topological space as a topological space in its own right, and it turns out that if X is a metric space, the topologies that one obtains on A in both fashions are identical.

Definition. If $(X, \mathbf{T})$ is a topological space and A is a subset of X, then the *subspace topology* on A is the family $\mathbf{T}|A$ of all intersections $U \cap A$ where $U \in \mathbf{T}$. — It is an elementary set-theoretic exercise to verify that this defines a topology on A.

The following result relates the metric and subspace topologies on a subset of a metric space.

PROPOSITION. *If X is a metric space and $A \subset X$, then the metric topology on A is identical to the subspace topology on A.*

Proof. It will be convenient to distinguish the open disks in A and X by N^A and N^X respectively. By construction we have that $N^A = A \cap N^X$.

Let $\mathbf{T}|A$ denote the subspace topology and let $\mathbf{M}_A$ denote the metric topology. Every set in $\mathbf{T}|A$ has the form $U \cap A$ where $U \in \mathbf{T}$, where $\mathbf{T}$ denotes the metric topology on X. By definition of the metric topology on X, for every point $y \in A \cap U$ there is an $r > 0$ such that $N_r^X(y) \subset U$, and therefore we have

$$N_r^A(y) = A \cap N_r^X(y) \subset A \cap U$$

which shows that every set in $\mathbf{T}|A$ belongs to $\mathbf{M}_A$. Conversely, every open set W in the metric topology is a union of open disks having the form $N_{r(y)}^A(y)$ for $y \in W$ and suitably chosen $r(y) > 0$ (use the lemma stated above), and therefore we have

$$W = \bigcup_y N_{r(y)}^A(y) = \bigcup_y \left(A \cap N_{r(y)}^X(y)\right) = A \cap \left(\bigcup_y N_{r(y)}^A(y)\right)$$

which shows that W is an intersection of A with an open subset of X.■

Neighborhoods

In a metric space the sets $N_\delta(x)$ are often called *δ-neighborhoods*; for general topological spaces one also uses the term *neighborhood of a point* (say) p to denote an open set containing p (compare the definition on page 96 of Munkres); however, sometimes the term "neighborhood" has a more general meaning of a set N such that N contains some open subset U which in turn contains the point p; one should be aware of this difference when reading other books.

II.2 : Closed sets and limit points

(Munkres, § 17)

The usual discussion of limits for sequences for real numbers extends directly to metric spaces. Given a metric space X and a sequence $\{a_n\}$ in X, we say that $\lim_{n\to\infty} a_n = a$ if for all $\varepsilon > 0$ there is a positive integer M such that $n > M$ implies $\mathbf{d}(a_n, a) < \varepsilon$. As in real variables, a sequence has at most one limit, and the proof in the general case is essentially the same.

Closed subsets of the real line are precisely the subsets that are "closed under taking limits of convergent sequences," so it is clear that one should be able to discuss closed subsets of an arbitrary metric space. At first glance it is less obvious that one can also discuss closed sets for arbitrary topological spaces, but this is indeed the case, and one major objective here to justify this.

Limit points and limits of sequences

We begin with the following definition:

Definition. Let X be a topological space, and let $A \subset X$ be a subset of X. A point $y \in X$ is called a *limit point* of A if for all open sets U containing y the intersection $A \cap (U - \{y\})$ is nonempty. The set of all limit points of A in X is written $\mathbf{L}(A; X)$, and when the ambient space X is clear from the context we shall often write $\mathbf{L}(A)$.

The motivation for the terminology is implicit in the following result:

PROPOSITION. *The following are equivalent for a metric space X, a point $y \in X$ and a subset $A \subset X$:*

(i) $y \in \mathbf{L}(A)$.

(ii) There is a sequence of points $\{a_n\}$ in A such that $a_n \neq y$ for all n but $\lim_{n\to\infty} a_n = y$.

Proof. $\big((i) \Longrightarrow (ii)\big)$ Let n be a positive integer, and consider the open set $N_{1/n}(y)$. By the definition of $\mathbf{L}(A)$ there is a point $a_n \in A$ such that $a_n \neq y$ but $a_n \in N_{1/n}(y)$. Then

$$\mathbf{d}(a_n, y) < \frac{1}{n}$$

for all n and therefore $\lim_{n\to\infty} a_n = y$.

$\big((ii) \Longrightarrow (i)\big)$ Let the sequence $\{a_n\}$ be given as in the statement of (ii), let U be an open subset containing y, let $\varepsilon > 0$ be such that $N_\varepsilon(y) \subset U$, and choose M such that $n > M$ implies $\mathbf{d}(a_n, a) < \varepsilon$. Then we have

$$a_{M+1} \in A \cap (N_\varepsilon(y) - \{y\}) \subset A \cap (U - \{y\})$$

and therefore $y \in \mathbf{L}(A)$.■

The nest result is the key to defining closed subsets in arbitrary topological spaces.

THEOREM. *If A is a subset of a topological space X, then $X - A$ is open if and only if* $\mathbf{L}(A) \subset A$.

Proof. $(\Longrightarrow)$ Suppose that $X - A$ is open and the set $\mathbf{L}(A)$ is not contained in A. Let $y \in \mathbf{L}(A) - A$; clearly $y \in X - A$. By the definition of the set of limit points, it follows that there is a point $x \in X - A$ that is also in A, which is a contradiction. Thus $\mathbf{L}(A) \subset A$ if $X - A$ is open.

($\Longleftarrow$) Suppose that $\mathbf{L}(A) \subset A$ and let $y \in X - A$. By hypothesis y is not a limit point of A and therefore there is some open set U_y containing y such that $U_y - \{y\}$ and A are disjoint (*Note:* You should check that the conclusion is the negation of the condition in the definition of limit point!). Since we also know that $y \notin A$ it follows that U_y and A are also disjoint, so that $U_y \subset X - A$. Therefore we have the string of inclusions

$$X - A \quad = \quad \bigcup_{y \notin A} \{y\} \quad \subset \quad \bigcup_{y \notin A} U_y \quad \subset \quad X - A$$

which shows that $X - A = \cup_y U_y$; since the right hand side is a union of open sets, it follows that the sets on both sides of the equation are open.■

This leads us to a topological definition of closed set that is compatible with the notion of "closure under limits of sequences" for metric spaces.

Definition. A subset F of a topological space X is *closed* if and only if its relative complement $X - A$ is open.

Note. In contrast with the usual usage for the terms "open" and "closed," a subset of a topological space may be open but not closed, closed but not open, neither open nor closed, or both open and closed. Over the real line these are illustrated by the subsets $(0,1)$, $[0,1]$, $[0,1)$ and $\mathbf{R}$ itself (you should verify this for each example).

Closed subsets have the following properties that correspond to the fundamental properties of open subsets.

PROPOSITION. *The family of closed subsets of a topological space X has the following properties:*

(i) *The empty set and X itself are both closed in X.*
(ii) *If for each $\alpha \in A$ the set F_α is closed in X then $\cap_\alpha F_\alpha$ is also closed in X.*
(iii) *If F_1 and F_2 are closed in X then $F_1 \cup F_2$ is also closed in X.*

Note. One can combine (iii) and finite induction to prove that the union of any finite collection of closed subsets in X is also closed in X.

Proof. (i) The empty set and X are complements of each other, so since each is open their complements — which are merely the empty set and X itself — are closed.

(ii) This follows immediately from the complementation formula

$$X - \bigcap_\alpha F_\alpha \quad = \quad \bigcup_\alpha (X - F_\alpha)$$

and the fact that unions of open subsets are open.

(iii) This follows immediately from the complementation formula

$$X - (F_1 \cup F_2) \quad = \quad (X - F_1) \cap (X - F_2)$$

and the fact that the intersection of two open subsets is open.

Clearly one could define mathematical systems equivalent to topological spaces by specifying families of closed subsets satisfying the three properties in the preceding proposition. In fact, there are also many other equivalent ways of describing topological spaces, but we shall not say very much about them here.■

For metric spaces one has the following important fact regarding closed subsets.

PROPOSITION. *If X is a metric space and $x \in X$, then the one point set $\{x\}$ is closed in X.*

This follows immediately from an earlier observation that $X - \{x\}$ is open in X if X is a metric space. As noted previously, the indiscrete topology on a set with at least two elements does not have the corresponding property.■

Closures and interiors of subsets

In many mathematical contexts it is useful and enlightening to have constructions that fill the gaps in a mathematical object. For example, the real numbers are a way of filling the gaps in the rational numbers. Given a subset of, say, the real line, one often wants to expand this set so that it contains all limits of sequences that are defined on that set. This is done by considering the *closure* of the set, and the concept can be formulated in a manner that applies to all topological spaces.

Definition. Given a topological space X and a subset $A \subset X$, the *closure* of A is the set $\overline{A} = A \cup \mathbf{L}(A)$.

The terminology suggests that $\overline{A}$ should be the smallest closed subset of X that contains A. Verifying this will take a little work.

PROPOSITION. *The set $\overline{A}$ is the intersection of all closed subsets containing A, and consequently it is the smallest closed subset containing A.*

Proof. Let F be the intersection described in the statement of the proposition. We need to prove the two inclusions $\overline{A} \subset F$ and $F \subset \overline{A}$.

$\left(\overline{A} \subset F\right)$ Since $A \subset F$ it is only necessary to show that $\mathbf{L}(A) \subset F$. It follows immediately from the definitions that $A \subset F$ implies $\mathbf{L}(A) \subset \mathbf{L}(F)$ (fill in the details here!). But since F is closed we know that $\mathbf{L}(F) \subset F$, and therefore we have that $\mathbf{L}(A) \subset F$ as required.

$\left(F \subset \overline{A}\right)$ This will follow if we can show that $\overline{A}$ is closed in X, or equivalently that $X - \overline{A}$ is open in X. So suppose that $y \in X - \overline{A}$. By definition this means that $y \notin A$ and $y \notin \mathbf{L}(A)$. The latter in turn means that there is an open set $U_y \subset X$ such that $A \cap (U_y - \{y\})$ is empty. But we also know that $y \notin A$,so we can strengthen the latter to say that $A \cap U_y$ is empty. We then have the usual chain of inclusions

$$X - \overline{A} \quad = \quad \bigcup_{y \notin \overline{A}} \{y\} \quad \subset \quad \bigcup_{y \notin \overline{A}} U_y \quad \subset \quad X - A$$

which shows that $X - \overline{A} = \cup_y U_y$ and therefore is open; but this means that $\overline{A}$ is closed.■

There is a complementary concept of the *interior* of a set A, which is the largest open subset U contained in A. Formally, one can define the interior by the formula

$$\mathrm{Int}\ (A) \quad = \quad X - \ \overline{X - A}$$

and the proof that this is the union of all open subsets contained in A reduces to an exercise in set theory.

For the sake of completeness, here are the details: One can rewrite the defining equation as $X - \mathrm{Int}\ (A) = \overline{X - A}$ and since the latter contains $X - A$, by taking complements we have that $\text{Int}(A)$ is an open set that is contained in A. Suppose now that U is any open subset contained in A. Then $X - U$ is a closed set that contains $X - A$ and thus also $\overline{X - A} = X - \text{Int}\,(A)$; taking complements once again we see that U is contained in $\text{Int}(A)$.■

Warning. In some topological contexts the term "interior" has a much different meaning, but in this course the term will always have the meaning given above.

The following result provides an extremely useful relation between the notions of closure and passage to subspaces.

PROPOSITION. *Given a topological space X and subspaces A, Y such that $A \subset Y \subset X$, let* $\text{Closure}_Y(A)$ *denote the closure of A with respect to the subspace topology on Y. Then*

$$\text{Closure}_Y(A) = \overline{A} \cap Y \ .$$

Proof. Once again we have to prove the inclusions in both directions.

$\big(\text{Closure}_Y(A) \subset \overline{A} \cap Y\big)$ Note first that the close subsets of Y have the form $Y \cap F$ where F is closed in X (*Proof:* E is closed in $Y \Leftrightarrow Y - E$ is open in $Y \Leftrightarrow Y - E = X \cap U$ for some U open in $X \Leftrightarrow E = Y - Y \cap U$ for some U open in $X \Leftrightarrow E = Y \cap (X - U)$ for some U open in X $\Leftrightarrow E = Y \cap F$ for some F closed in X). — It follows that the right hand side is a closed subset of Y and therefore contains the set $\text{Closure}_Y(A)$.

$\big(\overline{A} \cap Y \subset \text{Closure}_Y(A)\big)$ The right hand side is a closed subset of Y, and therefore by the preceding paragraph it has the form $B \cap Y$ where B is closed in X. By construction $B \supset A$, so B must also contain $\overline{A}$. But this means that

$$\overline{A} \cap Y \subset B \cap Y = \text{Closure}_Y(A)$$

which yields the desired inclusion.■

Convergence in general topological spaces

In general one cannot work with limits of sequences in abstract topological spaces as easily and effectively as one can work with them in metric spaces. The crucial property of metric spaces that allows one to work with sequences is the following:

First Countability Property. *If X is a metric space and $x \in X$ then there is a sequence of decreasing open subsets U_k such that every open subset contains some U_k.*

In fact, we can take U_k to be the open disk of radius $\frac{1}{k}$ centered at x (*i.e.*, $N_{1/k}(x)$).■

There is a somewhat more complicated concept of *net* that serve a similar purpose to sequences for arbitrary topological spaces. Nets for topological spaces are not as important or useful as sequences for metric spaces, but there are some situations where it is convenient to have them. A concise but readable introduction to nets appears on pages 187–188 of Munkres.

II.3 : Continuous functions

(Munkres, §§ 18, 21; Edwards, § I.8)

The standard definitions for continuous and uniformly continuous functions generalize immediately to metric spaces.

Definition. Let $(X, \mathbf{d}_X)$ and $(Y, \mathbf{d}_Y)$ be metric spaces, and let $a \in X$. A set-theoretic function $f : X \to Y$ is said to be *continuous* at a if for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $\mathbf{d_X}(x, a) < \delta$ implies $\mathbf{d_Y}(f(x), f(a)) < \varepsilon$. The function f is said to be continuous (on all of X) if it is continuous at every point of X.

As in the case of functions of a real variable, the numbers $\delta(\varepsilon)$ depend upon the point a.

Definition. Let $(X, \mathbf{d}_X)$ and $(Y, \mathbf{d}_Y)$ be metric spaces, and let $a \in X$. A set-theoretic function $f : X \to Y$ is said to be *uniformly continuous* if for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for all $u, v \in X$, we have that $\mathbf{d_X}(u, v) < \delta$ implies $\mathbf{d_Y}(f(u), f(v)) < \varepsilon$.

As in real variables, the difference between continuity and uniform continuity is that δ depends upon ε and a for continuity but it depends only upon ε for uniform continuity.

The following characterization of continuity yields a definition that is meaningful for functions on arbitrary topological spaces:

THEOREM. *Let $(X, \mathbf{d}_X)$ and $(Y, \mathbf{d}_Y)$ be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if for each open set $V \subset Y$, the inverse image $f^{-1}(V)$ is open in X.*

Proof. $(\Longrightarrow)$ Choose $y \in Y$ and $x \in X$ so that $y = f(x)$. There is an $\varepsilon > 0$ so that $N_\varepsilon(y) \subset V$, and by continuity there is a $\delta > 0$ such that f maps $N_\delta(x)$ into $N_\varepsilon(y)$. It follows as in many previous arguments that

$$f^{-1}(V) \quad = \quad \bigcup_x N_\delta(x)$$

(check this out!) and therefore the left hand side is an open subset of X.

$(\Longleftarrow)$ Choose $y \in Y$ and $x \in X$ so that $y = f(x)$, and let $\varepsilon > 0$ be given. By the hypothesis we know that the set

$$W = f^{-1}\left(N_\varepsilon(y)\right)$$

is an open subset of X containing x. If we choose $\delta > 0$ so that $N_\delta(x) \subset W$, then it follows that f maps $N_\delta(x)$ into $N_\varepsilon(y)$.■

In view of the above, if $(X, \mathbf{T}_X)$ and $(Y, \mathbf{T}_Y)$ are topological spaces we may **DEFINE** a set-theoretic map $f : X \to Y$ to be continuous if and only if for each open set $V \subset Y$, the inverse image $f^{-1}(V)$ is open in X.

Several equivalent formulations of continuity are established in Theorem 18.1 on pages 104–105 of Munkres and Lemma 21.3 on page 130 of Munkres. Here is an overlapping list of equivalences:

CHARACTERIZATIONS OF CONTINUITY. *Let X and Y be topological spaces and let $f : X \to Y$ be a set-theoretic map. then the following are equivalent:*

(1) *f is continuous.*

(2) *For every closed subset $F \subset Y$ the inverse image $f^{-1}(F)$ is closed.*

(3) *For all* $A \subset X$ *we have* $f(\overline{A}) \subset \overline{f(A)}$.

(4) *For all* $B \subset Y$ *we have* $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

(5) *For all* $A \subset X$ *we have* $\mathrm{Int}\,(f(A)) \subset f\,(\mathrm{Int}(A))$.

(6) *For all* $B \subset Y$ *we have* $f^{-1}\,(\mathrm{Int}(B)) \subset \mathrm{Int}\,\big(f^{-1}(B)\big)$.

If X *and* Y *are metric spaces then the following is also equivalent to the preceding conditions:*

(7) *For all sequences* $\{\, x_n \,\}$ *in* x *such that* $\lim_{n\to\infty} x_n = a$ *we have* $\lim_{n\to\infty} f(x_n) = f(a)$.

The statements and proofs of the results in Munkres should be read and understood. Verification of the statements not proven in Munkres is left to the reader as an exercise.■

It is not possible to discuss uniform continuity in a topological space unless some extra structure is added; one reference for an abstract treatment of such uniform structures (or uniformities) is Kelley, *General Topology*. Topological spaces with uniform structures are often known as *uniform spaces*. A class of spaces known as *topological groups* have particularly important examples of the uniform structures that exist on uniform spaces. An introduction to the theory of topological groups appears in Appendix A of these notes.

EXAMPLES. 1. A real variables textbook (and even a calculus or precalculus textbook) contains many examples of continuous functions from subsets of the real numbers to the real numbers.

2. If A is a subset of a topological space with the subspace topology, then the inclusion map $i : A \to X$ is continuous because $i^{-1}(U) = U \cap A$ for all open subsets U. In fact, if X is a metric space and A has the subspace metric, then the inclusion map is uniformly continuous; for each $\varepsilon > 0$ we can take $\delta = \varepsilon$.

3. An important special case of the preceding example occurs when $A = X$, and in this case the inclusion is the *identity map* on X.

4. Let X be an arbitrary metric space, and let A be a nonempty subset of X. For each point $x \in X$ the distance from x to A is defined by the formula

$$\mathbf{d}(x, A) = \mathrm{g.}\ell\mathrm{.b.}_{a\in A}\mathbf{d}(x, a)$$

where the greatest lower bound exists and is nonnegative because all distances are nonnegative. We claim that the function $\mathbf{d}(-, A)$ is uniformly continuous. — Here is a **Proof**: By the triangle inequality we have that $\mathbf{d}(x, a) \le \mathbf{d}(x, y) + \mathbf{d}(y, a)$ for all $x, y \in X$ and $a \in A$. Therefore it follows that $\mathbf{d}(x, A) \le \mathbf{d}(x, y) + \mathbf{d}(y, a)$. Subtract $\mathbf{d}(x, y)$ from each side. This yields the inequality $\mathbf{d}(x, A) - \mathbf{d}(x, y) \le \mathbf{d}(y, a)$, which in turn implies that the left hand side is $\le \mathbf{d}(y, A)$. We can now rewrite this in the form $\mathbf{d}(x, A) - \mathbf{d}(y, A) \le \mathbf{d}(x, y)$. If we reverse the roles of x and y in this argument we get the complementary inequality $\mathbf{d}(y, A) - \mathbf{d}(x, A) \le \mathbf{d}(x, y)$. Combining these, we obtain the inequality

$$\big|\mathbf{d}(y, A) - \mathbf{d}(x, A)\big| \le \mathbf{d}(x, y)$$

which shows that the function in question is in fact uniformly continuous because for each $\varepsilon > 0$ we can take $\delta = \varepsilon$.

5. We shall end this list of examples with one that is very simple. Suppose that X and Y are any topological spaces and that $y \in Y$. Then there is a constant map $C_y : X \to Y$ which sends every point of X to y. This map is continuous. To see this, let $V \subset Y$ be open, and consider $f^{-1}(V)$. If $y \in V$ then the inverse image is all of X but if $y \notin V$ then the inverse image is the empty set. In either case the inverse image is open.

In analysis there are theorems stating that sums, products and composites of continuous functions are continuous. Metric and topological spaces usually do not have the algebraic structure needed to construct sums and products. However, one does have the following version of continuity for composite functions.

PROPOSITION. *If X, Y, Z are topological spaces and $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is the composite $g \circ f : X \to Z$.*

Proof. Suppose that W is open in Z; then by continuity it follows that $V = g^{-1}(W)$ is open in Y and $U = f^{-1}(V)$ is open in X. However, we also have

$$u = f^{-1}\left(g^{-1}(W)\right) = (g \circ f)^{-1}(W)$$

and therefore it follows that $g \circ f$ is also continuous.■

COROLLARY. *If $f : X \to Y$ is continuous and $A \subset X$ is equipped with the subspace topology, then the restriction $F|A : A \to Y$ is continuous.*

This is true because the restriction is the composite of f and the inclusion map for $A \subset X$; we have already noted that the latter is continuous.

In addition to the preceding way of constructing continuous functions by restricting the domain, it is also possible to construct new continuous functions by shrinking the codomain if the image of the function is a proper subset.

PROPOSITION. *Let $f : X \to Y$ be a continuous function, let $B \subset Y$ be equipped with the subspace topology, let $j : B \to Y$ denote the inclusion map, and suppose that $f(X) \subset B$. Then there is a unique continuous map $g : X \to B$ such that $j \circ g = f$.*

Proof. On the set-theoretic level one simply defines g by the rule $g(x) = f(x)$. We need to verify that this map is continuous.

Suppose that V is open in B. Then $B = W \cap B$ where W is open in Y. Given an arbitrary subset $A \subset Y$, elementary set-theoretic considerations imply that

$$f^{-1}(A) = f^{-1}(A \cap B) = g^{-1}(A \cap B)$$

with the first equation holding because $f(X) \subset B$ and the second holding because $f(x) = g(x)$ for all x. Therefore if V is open in B and $V = W \cap B$ (where W is open in Y), then

$$g^{-1}(V) = f^{-1}(W \cap B) = f^{-1}(W) \ .$$

Since f is continuous the set on the right hand side of the equation is open in X; therefore the set on the left hand side is also open and the map g is continuous.■

Homeomorphisms and other special mappings

We begin with a natural question:

Continuity of inverses. *Suppose that $f : X \to Y$ is a continuous map of topological spaces that is a $1-1$ correspondence. Is the inverse map f^{-1} also continuous?*

There are many examples to show that the answer to the question is negative. One purely formal approach is to take a $X = Y$ with $f = \mathrm{id}_X$ and the topologies on the domain and codomain equal to the discrete and indiscrete topologies respectively. Then f is continuous (every map into a space with the indiscrete topology is continuous!). What can we say about the continuity of the inverse? By construction the inverse is just the identity map from a space with the indiscrete topology to a space with the discrete topology. If X has more than one element and A is a nonempty proper subset, then A is open in the discrete topology but not in the indiscrete topology, and therefore the inverse map is not continuous.

Here is a more tangible example. Let S^1 be the unit circle in the cartesian plane, and let $f : [0,1) \to S^1$ send t to $(\cos 2\pi t, \sin 2\pi t)$. Then f is clearly continuous and 1–1 onto (it might be helpful to draw a picture of this). However, f^{-1} is not continuous at the point $(1,0) \in \mathbf{R}^2$. Specifically, the set $[0, \frac{1}{2})$ is open in $[0,1)$, but its inverse image in the circle under f^{-1} — which is simply $f\left([0, \frac{1}{2})\right)$ (why?) — is not open in the circle. To see this, note that every open subset of the circle containing $(1,0)$ must have some points whose second coordinates are negative.

We are thus led to the following:

Fundamental Definition. A continuous 1–1 onto map $f : X \to Y$ of topological spaces is a *homeomorphism* if f^{-1} is also continuous.

It follows immediately from the definition that for every topological space X the identity map id_X is continuous (it is understood that X has the same topology whether it is viewed as the domain or the codomain), the inverse of a homeomorphism is a homeomorphism, and the composite of two (composable) homeomorphisms is a homeomorphism.

Here is an alternate characterization of homeomorphisms that may be enlightening:

PROPOSITION. *Let $(X, \mathbf{T}_X)$ and $(Y, \mathbf{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a set-theoretic map that is $1-1$ and onto. Then f is a homeomorphism if and only if for each subset $A \subset X$, we have that A is open in X if and only if $f(A)$ is open in Y.*

Proof. $(\Longrightarrow)$ Let $\subset X$. If $f(A)$ is open in Y, then by continuity of f we have that

$$A = f^{-1}\left(f(A)\right)$$

is open in X. Similarly, if A is open in X, then by continuity of f^{-1} we have that

$$f(A) = [f^{-1}]^{-1}(A)$$

is open in Y.

$(\Longleftarrow)$ The hypotheses on f and the first set-theoretic identity in the previous paragraph imply that f is continuous, and the hypotheses together with the second set-theoretic identity in the previous paragraph imply that f^{-1} is continuous.∎

One can state and prove a similar theorem in which "open" is replaced by "closed" (in fact, the argument is essentially the same with this substitution).

The preceding result and its analog for closed sets lead to some other important classes of mappings on topological spaces.

Definitions. A set-theoretic map $f : X \to Y$ of topological spaces is *open* if for each open set $U \subset X$, the image $f(U)$ is open in Y. Similarly, a set-theoretic map $f : X \to Y$ of topological spaces is *closed* if for each closed set $A \subset X$, the image $f(A)$ is closed in Y.

Here are some instructive examples:

1. The identity map from a set with the indiscrete topology to a set with the discrete topology is both open and closed but not continuous.

2. The previously constructed map from $[0, 1)$ to S^1 is continuous but neither open nor closed. Its inverse is open and closed but not continuous.

3. The map from $\mathbf{R}^2$ to $\mathbf{R}$ sending (x, y) to x is continuous and open but not closed. [*Hints:* The proof that the map is open reduces to showing that the image of an open δ-disk is always open. Why is the image of the open δ-disk about (x, y) equal to the open interval $\{t \mid x - \delta < t < x + \delta\}$? To show the map is not closed consider the graph of $1/x$ and its image under the given map.]

4. The map from $\mathbf{R}$ to itself sending x to its absolute value is continuous and closed but not open. [*Hints:* To show the map is not open, consider the image of the whole space. To show it is closed, explain why every closed subset F can be written as a union $F = F_+ \cup F_-$ where $F_\pm$ is a closed set consisting of all points in F that are respectively nonnegative or nonpositive. Why does f take $F_\pm$ to a closed set, and how does this show that f is closed?]

The following result is elementary:

PROPOSITION. *The composite of two open mappings is open, and the composite of two closed mappings is closed. Identity maps are always open and closed.*■

For metric spaces there is an extremely special type of homeomorphism:

Definition. Let $(X, \mathbf{d}_X)$ and $(Y, \mathbf{d}_Y)$ be metric spaces. A set-theoretic function $f : X \to Y$ is said to be an *isometry* if it is onto and $\mathbf{d}_X(u, v) = \mathbf{d}_Y(f(u), f(v))$ for all $u, v \in X$. — Such a map is automatically 1–1 because

$$u \neq v \implies \mathbf{d}_Y(f(u), f(v)) = \mathbf{d}_X(u, v) > 0 \ .$$

By construction such a map is also uniformly continuous and has a uniformly continuous inverse (which is also an isometry). Note that identity maps are always isometries, the composites of isometries are isometries, and inverses to isometries are isometries.

Different metrics determining the same topology

The discussion of diameters and bounded metric spaces on pages 121–122 of Munkres should be read at this point. Theorem 20.1 on page 121 is an important case of an extremely general phenomenon: *If $(X, \mathbf{d})$ is a metric space, then in general there are many different metrics $\mathbf{e}$ such that the identity map from $(X, \mathbf{d})$ to $(X, \mathbf{e})$ is a homeomorphism. Furthermore, in general there are many examples for which the identity map is also uniformly continuous, and in fact one can even find large classes of examples for which the identity map in the opposite direction is also uniformly continuous.* — Important examples of this will arise later in the course.

Metric spaces of functions

We have already noted that the set $\mathbf{BF}(X)$ of bounded functions on a set X has a metric space structure with

$$\mathbf{d}(f, g) = \sup_x \left| f(x) - g(x) \right| \ .$$

In such a space the limit of a sequence of functions corresponds to uniform convergence: We have $\lim_{n\to\infty} f_n = f$ if and only if for all $\varepsilon > 0$ we can find M so that $n > M$ implies $|f_n(x) - f(x)| < \varepsilon$

for all $x \in X$ (this is not quite trivial because the latter inequality only implies $|f_n - f| \le \varepsilon$, but if the condition holds we can also find M' so that $n > M'$ implies $|f_n(x) - f(x)| < \varepsilon/2$ and the latter certainly implies $|f_n - f| \le \varepsilon/2 < \varepsilon$), A large amount of the theory of uniform convergence for functions of real variables carries over to this general setting. In particular, pages 147–151 of Rudin, *Principles of Mathematical Analysis (Third Edition)*, go through with only minor changes. In particular, at certain points in a point set topology course it is necessary to use the following result, which is proved on page 132 of Munkres or pages 149–150 of the book by Rudin mentioned above.

THEOREM. *Let X be a topological space, let $\mathbf{BF}(X)$ be defined as above, and let $\mathbf{BC}(X)$ denote the set of all continuous functions in $\mathbf{BF}(X)$. Suppose that $\lim_{n\to\infty} f_n = f$ in $\mathbf{BF}(X)$ where each of the functions f_n is continuous. Then f is also continuous.*■

The spaces $\mathbf{BC}(X)$ also have a great deal of algebraic structure (for example, addition and multiplication of functions) that one has for continuous functions on, say, the unit interval. This is all discussed in Section 21 of Munkres (pp. 129–133).

Piecing together continuous functions

The previous material provides one powerful method for constructing continuous functions on metric and topological spaces. This is essentially an analytic method for constructing functions. It is often important to have a similarly useful geometric method for obtaining continuous functions.

Geometric piecing problem. *Suppose that we are given topological spaces X and Y, a family of subsets $\{A_\alpha\}$ of X and a continuous function $f_\alpha : A_\alpha \to Y$ for each α. Is it possible to form a continuous function $g : \cup_\alpha A_\alpha \to Y$ such that $g(x) = f_\alpha(x)$ if $x \in A_\alpha$?*

Notation. Given a function $h : A \to B$ and $C \subset B$ we shall denote the composite of h with the inclusion $C \to A$ by $h|C$ and call it the *restriction of h to C*. Note that if h is a continuous map of topological spaces and C has the subspace topology then $h|C$ is automatically continuous.

There is an obvious set-theoretic condition that is necessary if a function g as above actually exists. Namely, for all $y \in A_\alpha \cap A_\beta$ we need the consistency condition $f_\alpha(y) = f_\beta(y)$. In terms of the restriction notation this can be rewritten formally as

$$f_\alpha|\, A_\alpha \cap A_\beta = f_\beta|\, A_\alpha \cap A_\beta .$$

For certain families of subspaces this turns out to be the only condition needed to piece together a continuous function defined on an entire space.

THEOREM. *Let X and Y be topological spaces, let $\mathcal{A} = \{A_\alpha\}$ be a family of subsets of X such that $X = \cup_\alpha A_\alpha$, and for each let $f_\alpha : A_\alpha \to Y$ be a continuous function. Assume that these functions satisfy the consistency condition $f_\alpha|\, A_\alpha \cap A_\beta = f_\beta|\, A_\alpha \cap A_\beta$. If either*

(i) $\mathcal{A}$ is a family of open subsets,

(ii) $\mathcal{A}$ is a **finite** *family of closed subsets,*

then there is a unique continuous function $f : X \to Y$ such that $f|A_\alpha = f_\alpha$ for all α.

To see that a similar result does not hold for arbitrary families of closed subsets, consider the family $\mathcal{A}$ of one point sets $\{x\}$. Given an arbitrary set-theoretic function g from a metric space X to a topological space Y, the restrictions $g|\{x\}$ are all continuous and the consistency condition follows because the sets in the family are pairwise disjoint. Thus any discontinuous function g

satisfies the conditions for the family of closed subsets $\mathcal{A}$, and for most choices of X there are many discontinuous functions to choose from.

Proof of Theorem. *First case.* The consistency condition ensures that there is a well-defined set-theoretic function $f : X \to Y$ with the desired properties, so the real issue is to prove this function is continuous. Let V be an open subset of Y. Then we have

$$f^{-1}(V) = \bigcup_{\alpha} \left(A_\alpha \cap f^{-1}(V)\right)$$

and since $(f|A_\alpha)^{-1}(V) = A_\alpha \cap f^{-1}(V)$ the right hand side of the displayed expression is simply the union of the sets $(f|A_\alpha)^{-1}(V) = f_\alpha^{-1}(V)$. By the continuity of the functions f_α the sets on the right hand side of this expression are open, and therefore the union of these sets, which is just $f^{-1}(V)$, is also open, proving that f is continuous.

Second case. Many of the steps in the argument are the same so we shall concentrate on the differences. First of all, we need to replace the open subset V with a closed subset F. The same argument then shows that

$$f^{-1}(F) = \bigcup_{\alpha} f_\alpha^{-1}(F)$$

where each summand on the right hand side is closed by continuity. Since the union on the right hand side is finite, the union on the right hand side is again a closed subset, and this implies that $f^{-1}(F)$ is closed in X.■

II.4 : Cartesian products

(Munkres, §§ 15, 19)

Product constructions are useful in mathematics both as a means of describing more complicated objects in simpler terms (for example, expressing vectors in terms of magnitude and direction or resolution into x, y and z components) and also as the basis for considering quantities (formally, functions) whose values depend upon several variables.

Topological structures on finite products

Since product structures are less ambiguously defined for topological spaces as opposed to metric spaces, we shall begin with the former.

There are two ways of viewing Cartesian products with finitely many factors. Clearly one wants the product of the sets $A_1, \cdots, A_n$ to be the set of all ordered n-tuples (or lists of length n) having the form $(a_1, \cdots, a_n)$ where $a_i \in A_i$ for all i. Formally these can be described as functions α from $\{1, \cdots, n\}$ to $\cup_i A_i$ such that $\alpha(i) \in A_i$ for all i. Alternatively, one can view finite products as objects constructed inductively from 2-fold products by the recursive formula

$$A_1 \times \cdots \times A_{n+1} = (A_1 \times \cdots \times A_n) \times A_{n+1}$$

for all $n \geq 2$. It is an elementary exercise to see that the two formulations both result in equivalent concepts of ordered n-tuples with the property

$$(a_1, \cdots, a_n) = (b_1, \cdots, b_n) \iff a_i = b_i, \qquad \forall\, i\ .$$

Definition. Let $n \geq 2$ be an integer, and for each integer i between 1 and n let $(X_i, \mathbf{T}_i)$ be a topological space. The *product topology* on $X_1 \times \cdots \times X_n$ is the topology generated by all sets of the form $U_1 \times \cdots \times U_n$, where $U_i \in \mathbf{T}_i$ for all i. Frequently we shall write $\prod_i X_i$ to denote the product of the sets X_i and $\prod_i (X_i, \mathbf{T}_i)$ to denote the product topology; if the topologies on the factors are clear from the context and it is also clear that we want the product topology on $\prod_i X_i$ we shall frequently use the latter to denote the product space.

Before proceeding further we make a simple but useful observation.

PROPOSITION. *Every open subset in the product topology is a union of open subsets of the form $U_1 \times \cdots \times U_n$ where $U_i \in \mathbf{T}_i$ for all i.*

Proof. The topology generated by a family $\mathcal{F}$ of subsets consists of arbitrary unions of finite intersections of sets in $\mathcal{F}$, so it suffices to show that the latter is closed under finite intersections; by associativity and induction it suffices to check this for the intersections of pairs of subsets. But if we are given $\prod_i V_i$ and $\prod_i W_i$ where V_i and W_i are open in X_i for all i, then se have

$$\prod_i V_i \ \bigcap\ \prod_i W_i \quad = \quad \prod_i (V_i \cap W_i)$$

so that the family of products of open subsets is closed under finite intersections.

COROLLARY. *Let $\mathcal{B}_i$ be a base for the topology on X_i. Then every open subset in the product topology is a union of open subsets of the form $V_1 \times \cdots \times V_n$ where $V_i \in \mathcal{B}_i$ for all i.*

Proof. Since a union of unions is a union, it suffices to show this for open subsets of the form $U_1 \times \cdots \times U_n$, where $U_i \in \mathbf{T}_i$ for all i. Since each $\mathcal{B}_i$ is a base for $\mathbf{T}_i$, we can express each U_i as a union $\cup_{\alpha[i]} V_{\alpha[i]}$, and it follows that $U_1 \times \cdots \times U_n$ is equal to

$$\left(\bigcup_{\alpha[1]} V_{\alpha[1]}\right) \times \cdots \times \left(\bigcup_{\alpha[n]} V_{\alpha[n]}\right) =$$

$$\bigcup_{(\alpha[1], \ldots, \alpha[n])} \prod_i V_{\alpha[i]}$$

and hence $U_1 \times \cdots \times U_n$ is a union of products of the prescribed type.∎

The following result provides some motivation for the definition:

THEOREM. *For each $n \geq 2$ the topology on $\mathbf{R}^n$ with respect to the Euclidean metric is equal to the product topology associated to the family $(X_i, \mathbf{T}_i)$, where each space is the real line with the usual topology.*

Proof. *First step — Open sets in the product topology are open in the metric topology.* By the preceding corollary, every open subset in the product topology is a union of sets of the form $U_1 \times \cdots \times U_n$ where each U_i is an open interval in $\mathbf{R}$, and since arbitrary unions of metrically open sets are metrically open, it suffices to show that each product of open intervals $\prod_i (a_i, b_i)$ is open in the metric topology. Let $x = (x_1, \cdots, x_n)$ be a point in this product, and let $\varepsilon > 0$ be such that

$$\varepsilon \quad < \quad (x_i - a_i), \quad (b_i - x_i)$$

for all i. If $y = (y_1, \cdots, y_n)$ satisfies $\mathbf{d}(y, x) < \varepsilon$ with respect to the standard Euclidean metric then we have

$$|y_i - x_i| \quad \leq \quad \mathbf{d}(y, x) \quad < \quad \varepsilon$$

for all i. It is an elementary exercise to check that this displayed inequality and the previous one imply $y_i \in (a_i, b_i)$ for all i. Therefore $\prod_i (a_i, b_i)$ is open in the metric topology because $N_\varepsilon(x)$ is contained in this product.

Second step — Open sets in the metric topology are open in the product topology. It suffices to show that for each $x \in \mathbf{R}^n$ and $\varepsilon > 0$ there is a $\delta > 0$ so that $\prod_i (x_i - \delta, x_i + \delta)$ is contained in $N_\varepsilon(x)$. To see this, note that given a metrically open subset U we may write it as

$$\bigcup_{x \in U} N_{\varepsilon(x)}(x)$$

for suitable positive real numbers $\varepsilon(x)$. The latter union then contains the union

$$\bigcup_{x \in U} \prod_i (x_i - \delta(x, \varepsilon), x_i + \delta(x, \varepsilon))$$

which in turn contains $\cup_x \{x\} = U$. Thus U is a union of sets having the form

$$\prod_i (x_i - \delta, x_i + \delta)$$

and hence is open in the product topology.

The proof of the assertion is best understood using a simple picture in the plane. Consider the open disk in the uv-plane consisting of all points for which $u^2 + v^2 < 1$. How large of an open square centered at the origin can one fit inside this open disk? In particular, one can ask this for a square whose sides are parallel to the coordinate axes. It turns out that the square in question has one vertex of the form $(1/\sqrt{2}, 1/\sqrt{2})$ and the other three vertices given by multiplying either or both coordinates of the latter by -1. Now suppose we are looking inside the open unit disk in coordinate 3-space. What is the largest cube in that case? The coordinates of the vertices turn out to be $\pm 1/\sqrt{3}$. One can then form an educated guess regarding the vertices for a maximal n-dimensional hypercube inside the unit n-dimensional hyperdisk.

Formally, proceed as follows. Given a fixed n and an arbitrary $\varepsilon > 0$, let

$$\delta = \frac{\varepsilon}{\sqrt{n}}$$

and consider the set $\prod_i (x_i - \delta, x_i + \delta)$. If y belongs to this set then $|y_i - x_i| < \delta$ for all i and therefore

$$\mathbf{d}(x,y) \quad = \quad \left(\sum_i |x_i - y_i|^2\right)^{1/2} \quad < \quad \left(\sum_i \frac{\varepsilon^2}{n}\right)^{1/2} \quad = \quad \varepsilon$$

as required.■

General properties of (finite) product topologies

Given a sequence of sets $X_1, \cdots, X_n$ and an integer j between 1 and n, there is a map

$$p_j : \ \prod_i X_i \to X_j$$

called *projection onto the* i^{th} *coordinate* defined by the formula

$$p_j(x_1, \ \cdots \ , x_n) = x_j \ .$$

The following result characterizes the product topology in terms of these projections:

PROPOSITION. *Let $(X_i, \mathbf{T}_i)$ be a topological space for $1 \leq i \leq n$, and let $\prod_i \mathbf{T}_i$ denote the product topology on the product $\prod_i X_i$. Then $\prod_i \mathbf{T}_i$ is the unique smallest topology such that each projection map p_j is continuous.*

Proof. *Continuity of projections.* Let W be open in X_j. Then

$$p_j^{-1}(W) = \prod_i W_i$$

where $W_i = X_i$ if $i \neq j$ and $W_j = W$. This product set is open in the product topology and therefore p_j is continuous.

Minimality property. Suppose that $\mathbf{T}$ is a topology on $\prod_i X_i$ such that each p_j is continuous. Let $U = \prod_i U_i$ where U_i is open in X_i; we need to show that U is open with respect to $\mathbf{T}$. By the continuity of the projections we know that each set $p_j^{-1}(U_j)$ is open with respect to **T**, and therefore the finite intersection

$$\bigcap_i p_i^{-1}(U_i) \;=\; \prod_i U_i$$

is open with respect to **T**. Since every open set in the product topology is a union of sets of the form $\prod_i U_i$ it follows that every open set in the product topology is also open with respect to **T**.■

COROLLARY. *Let $(X_i, \mathbf{T}_i)$ be a topological space for $1 \leq i \leq n$, let $(Y, \mathbf{W})$ be a topological space, for each i let $f_i : Y \to X_i$ be a set-theoretic function, and let $f : Y \to \prod_i X_i$ be the unique function such that $f \circ p_i = f_i$ for all i so that $f(y) = (f_1(y), \cdots, f_n(y))$. Then f is continuous (with respect to the product topology on $\prod_i X_i$) if and only if each function f_i is continuous.*

Proof. If f is continuous then the continuity of the projections p_i and the continuity of composites imply that each f_i is continuous because $f_i = p_i \circ f$.

Now suppose that each f_i is continuous. If we can show that the inverse image of each basic open subset $\prod_i U_i$ is under f is open, then since inverse images preserve unions it will follow that the inverse image of every open set under f is open and hence that f is continuous. As before we know that

$$\prod_i U_i = \bigcap_i p_i^{-1}(U_i)$$

and if we take inverse images (and use the fact that inverse images preserve intersections) then we have

$$f^{-1}\left(\prod_i U_i\right) = \bigcap_i f^{-1} \circ p_i^{-1}(U_i) = \bigcap_i f_i^{-1}(U_i)$$

and the latter is open because each f_i is continuous.■

By construction a product of open subsets is open in the product topology (where we are only dealing with finite products). The analogous statement for closed subsets is also true:

PROPOSITION. *Let $(X_i, \mathbf{T}_i)$ be a topological space for $1 \leq i \leq n$, and for each i suppose that F_i is a closed subset of X_i. Then $\prod_i F_i$ is a closed subset of $\prod_i X_i$ with respect to the product topology.*

Proof. This follows from the set-theoretic equation

$$\bigcap_i p_i^{-1}(F_i) \;=\; \prod_i F_i$$

the continuity of the projections p_i and the fact that inverse images of closed subsets with respect to a continuous function are closed.■

COROLLARY. *Let $(X_i, \mathbf{T}_i)$ be a topological space for $1 \leq i \leq n$, and for each i suppose that A_i is a subset of X_i. Then*

$$\prod_i \overline{A_i} \;=\; \overline{\prod_i A_i} \;.$$

Proof. The first set in the display contains the second because the first is a closed set containing the product of the A_i and the second is the smallest such closed subset. To see that the first is contained in the second, let b be a point in the product of the closure, and let U be an open subset of $\prod_i X_i$ that contains b; we need to prove that $U \cap \prod_i A_i \neq \emptyset$. Let $\prod_i V_i$ be a basic open subset that contains b and is contained in U. Since the coordinates of b satisfy $b_j \in \overline{A_j}$ for all j, it follows that $V_j \cap A_j \neq \emptyset$ for all j, and from this we have that $\prod_i V_i \cap \prod_i A_i \neq \emptyset$; since $U \supset \prod_i V_i$ it also follows that $U \cap \prod_i A_i \neq \emptyset$. But this means that b lies in the closure of $\prod_i A_i$.■

Projection maps also have the following important property.

OPENNESS OF PROJECTIONS. *The coordinate projection maps* $p_j : \prod_i X_i \to X_j$ *are open.*

Proof. The set-theoretic equality

$$g\left(\bigcup W_\alpha\right) \quad = \quad \bigcup_\alpha g(W_\alpha)$$

shows that it suffices to prove $p_j(W)$ is open if W is a basic open subset. But such a set has the form $\prod_i U_i$ where each U_i is open in X_i, and the image of this set under p_j is simply W_j.■

We have already given an example to show that coordinate projections are not necessarily closed; namely projection onto either coordinate is a continuous and open map from $\mathbf{R}^2$ to $\mathbf{R}$, but the image of the closed set of points satisfying the equation $xy = 1$ (geometrically a hyperbola whose asymptotes are the x- and y-axes) is $\mathbf{R} - \{0\}$, which is not a closed subset of the real line.

Products and morphisms

If we are given a sequence of set-theoretic functions $f_i : X_i \to Y_i$, then there one can define the *Cartesian product of morphisms*

$$F = \prod_i f_i \; : \; \prod_i X_i \quad \to \quad \prod_i Y_i$$

by the formula

$$F(x_1, \; \cdots \; , x_n) = (f_1(x_1), \; \cdots \; , f_n(x_n))$$

or alternatively by the conditions

$$\pi_i^Y \circ F = f_i \circ \pi_i^X$$

where π_i^X and π_i^Y denote the i^{th} coordinate projections for $\prod_i X_i$ and $\prod_i Y_i$ respectively. Maps of this sort arise very frequently when one constructs new continuous functions out of old ones. If $n = 2$ one often describes such product maps using notation of the form

$$f_1 \times f_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$$

and similar notation is often used for other small values of n. Here are some properties of the product construction that are extremely elementary but also extremely important in many situations:

PROPOSITION. (*i*) *In the preceding notation, if each* X_i *and* Y_i *are topological spaces the function* F *is continuous with respect to the product topologies if and only if each* f_i *is continuous.*

(*ii*) *If each* f_i *is an identity map, then so is* F.

(*iii*) *Suppose we are also given sets* Z_i *and (set-theoretic) maps* $g_i : Y_i \to Z_i$, *and we set* G *equal to* $\prod_i g_i$. *Then*

$$G \circ F \quad = \quad \prod_i (g_i \circ f_i) \; .$$

The verifications of these statements are left to the reader as exercises.

Another important class of morphisms involving products are the maps that permute coordinates. We shall only discuss the simplest example here. Given two topological spaces X_1 and X_2 the *twist map*

$$\tau(X_1, X_2) : X_1 \times X_2 \longrightarrow X_2 \times X_1$$

is the map sending $(a, b) \in X_1 \times X_2$ to $(b, a) \in X_2 \times X_1$. These maps have the following elementary but important properties:

PROPOSITION. *If X_1 and X_2 are topological spaces, then $\tau(X_1, X_2)$ is continuous with respect to the product topologies. In fact, it is a homeomorphism whose inverse is given by $\tau(X_2, X_1)$.*

Proof. The second statement is purely set-theoretic and is elementary to verify. To check the continuity of the twist map, let p_1 and p_2 be the coordinate projections for the domain and let q_1 and q_2 be the coordinate projections for the codomain. We then have the identities

$$q_1 \circ \tau(X_1, X_2) \;=\; p_2 \,. \qquad q_2 \circ \tau(X_1, X_2) \;=\; p_1$$

and the continuity of $\tau(X_1, X_2)$ follows immediately from these.

Products and metric spaces

If $(X_i, \mathbf{d}_i)$ are metric spaces for $1 \leq i \leq n$, then it is possible to put metrics on $\prod_i X_i$ whose underlying topologies are the product topology. In fact, there are three particularly important product metrics. We shall describe three specific examples that are particularly significant. Let $x, y \in \prod_i X_i$ and express them in terms of coordinates as $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ respectively. Then the following formulas define metrics on the product:

$\mathbf{d}^{\langle\infty\rangle}(x, y) = \max_i\{\ \mathbf{d}_i(x_i, y_i)\ \}$.

$\mathbf{d}^{\langle 2\rangle}(x, y) = \left(\sum_i \mathbf{d}_i(x_i, y_i)^2\right)^{1/2}$.

$\mathbf{d}^{\langle 1\rangle}(x, y) = \sum_i \mathbf{d}_i(x_i, y_i)$.

The verification that each formula defines a metric is left to the reader as an exercise. We then have the following result:

PROPOSITION. *The topology determined by the metric $\mathbf{d}^{\langle\infty\rangle}$ is the product topology. Furthermore, the identity map from $(\prod_i X_i, \mathbf{d}^{\langle\alpha\rangle})$ to $(\prod_i X_i, \mathbf{d}^{\langle\beta\rangle})$ is uniformly continuous for all choices of $\alpha, \beta \in \{1, 2, \infty\}$.*

Proof. To verify the assertion about $\mathbf{d}^{\langle\infty\rangle}$ note that

$$\mathbf{d}^{\langle\infty\rangle}(x, y) < \varepsilon \iff y_i \in N_\varepsilon(x_i),\ \forall i \,.$$

Thus the ε-neighborhood of x with respect to the $\mathbf{d}^{\langle\infty\rangle}$ metric is just

$$\prod_i N_\varepsilon(x_i) \,.$$

By previous results, a base for the product topology on $\prod_i X_i$ is given by open sets of the form

$$\prod_i N_{\varepsilon(i)}(x_i)$$

and this implies that the product topology contains the metric topology. On the other hand, for each j the projection map $p_j : \prod_i X_i \to X_j$ is uniformly continuous because

$$\mathbf{d}_j\left(p_j(x), p_j(y)\right) \quad = \quad \mathbf{d}_j(x_j, y_j) \quad \leq \quad \mathbf{d}^{\langle\infty\rangle}(x, y)$$

implies we can take $\delta = \varepsilon$ in the criterion for uniform continuity. This means that the metric topology contains the product topology, and therefore by the previous observations we see that the topologies are equal.

The uniform continuity statements are direct consequences of the following inequalities for nonnegative real numbers α_i for $1 \leq i \leq n$:

$$\max_i\{\ \alpha_i\ \} \quad \leq \quad \left(\sum_i \alpha_i^2\right)^{1/2} \quad \leq \quad \sum_i \alpha_i \quad \leq \quad n \cdot \max_i\{\ \alpha_i\ \}$$

The middle inequality is perhaps the least trivial, and it can be verified by squaring both sides and noting that the corresponding inequality holds for the squares. These inequalities imply that the identity maps

$$\left(\prod_i X_i, \mathbf{d}^{\langle\infty\rangle}\right) \to \left(\prod_i X_i, \mathbf{d}^{\langle 1\rangle}\right) \to \left(\prod_i X_i, \mathbf{d}^{\langle 2\rangle}\right) \to \left(\prod_i X_i, \mathbf{d}^{\langle\infty\rangle}\right)$$

are uniformly continuous (and in fact the δ corresponding to a given ε can be read off explicitly from the inequalities!), and of course the composites of any two or three consecutive maps from this diagram are also uniformly continuous.■

The following basic result on products and metric spaces is also worth mentioning:

PROPOSITION. *If X is a metric space then the distance function $\mathbf{d} : X \times X \to \mathbf{R}$ is continuous (where $\mathbf{R}$ has the usual topology).*

Proof. Let $(x, y) \in X \times X$, and view the product topology as coming from the maximum metric by the preceding discussion. Given $\varepsilon > 0$ suppose that $(u, v) \in X \times X$ satisfies

$$\max(\ \mathbf{d}(x, u),\ \mathbf{d}(y, v)\ ) \ < \ \frac{\varepsilon}{2}\ .$$

Then several applications of the triangle inequality show that $\mathbf{d}(u, v) - \mathbf{d}(x, y) \ < \ \varepsilon$ and therefore $\mathbf{d}$ is in fact *uniformly continuous.*

Products and the Hausdorff Separation Property

We shall say that a topological space X has the *Hausdorff Separation Property* (or more simply, it is Hausdorff) if for each pair of distinct points $u, v \in X$ there are disjoint open subsets $U, V \subset X$ such that $u \in U$ and $v \in V$. As noted before, metric spaces have this property but it does not necessarily hold for an arbitrary topological space.

PROPOSITION. *In a Hausdorff space every one point subset is closed.*

Proof. Given $p \in X$ we shall show that $X - \{p\}$ is open if X is Hausdorff. Suppose that $y \in X - \{p\}$. Then there are disjoint open subsets U_y and V_y such that $p \in U_y$ and $y \in V_y$. Therefore we have

$$X - \{p\} \quad = \quad \bigcup_{y \neq p} \{y\} \quad \subset \quad \bigcup_{y \neq p} V_y \quad \subset \quad X - \{p\}$$

which implies that the last two subsets are equal, and thus $X - \{p\}$ is open because it is a union of open subsets.■

We now come to a result that has appeared on countless examinations:

THEOREM. *Given a set X, let the diagonal δ_X denote the set of all points $(u, v) \in X \times X$ such that $u = v$. Then X is Hausdorff if and only if Δ_X is closed in $X \times X$ with respect to the product topology.*

Proof. This follows because each of the statements listed below is equivalent to the adjacent one(s):

(1) X is Hausdorff.

(2) Given $(u, v) \in X \times X - \Delta_X$ there are open subsets $U, V \subset X$ such that $u \in U$, $v \in V$ and $\big(U \times V\big) \cap \Delta_X = \emptyset$.

(3) $X \times X - \Delta_X$ is open in $X \times X$ with respect to the product topology.

(4) Δ_X is closed in $X \times X$ with respect to the product topology.

Taken together, these prove the result.■

The theorem has an extremely important consequence:

PROPOSITION. *Let Y be a Hausdorff space, and let f and g be continuous functions from a topological space X to Y. Then the set*

$$E = \{\ x \in X \mid f(x) = g(x)\ \}$$

is closed in X.

Proof. If $H : X \to Y \times Y$ is the function defined by

$$H(x) = (f(x), g(x))$$

then we have already noted that H is continuous if f and g are continuous. It follows immediately that $E = H^{-1}(\Delta_Y)$. Since Y is Hausdorff we know that Δ_Y is closed in $Y \times Y$ and therefore its inverse image under H, which is simply E, is closed in X.■

SPECIAL CASE. *If f and g are continuous real valued functions on the unit interval* $[0, 1]$ *and $f(x) = g(x)$ for all rational points of* $[0, 1]$*, then $f = g$.*

Proof. The proposition shows that if f and g are continuous functions from the same space X into a Hausdorff space Y and $f|A = g|A$ then $f|\overline{A} = g|\overline{A}$. In this case A is the set of all rational points in $X = [0, 1]$ and $\overline{A} = X$. More generally this argument shows that if Y is a Hausdorff space, X is any space and $A \subset X$ is a subspace such that $\overline{A} = X$ and $f|A = g|A$, then $f = g$.■

It is easy to construct counterexamples to the conclusion of the proposition if the codomain is not Hausdorff. Suppose that X and Y both have the associated indiscrete topologies where both sets have at least two elements. Then every function from X to Y is continuous, and every nonempty subset $A \subset X$ is dense (*i.e.*, $\overline{A} = X$). Large families of counterexamples can be constructed in this manner; details are left to the reader as an exercise.

Infinite products

In earlier decades infinite products of topological spaces received a great deal of attention. We shall not deal with such objects extensively here, but it seems worthwhile to say a little about them for the sake of completeness and to avoid some natural possibilities for misunderstandings.

Given an indexed family of sets X_α with indexing set A, the set-theoretic cartesian product

$$\prod_{\alpha \in A} X_\alpha$$

may be defined formally as the set of all set-theoretic functions x from A to $\cup_\alpha \ X_\alpha$ such that $x(\alpha) \in X_\alpha$ for all α. This captures the intuitive ideas that the elements of the cartesian product are given by the coordinates x_α and that two elements are equal if and only if all their coordinates are equal.

The Axiom of Choice in set theory is equivalent to the statement that *if each of the sets X_α is nonempty, then so is their cartesian product $\prod_\alpha X_\alpha$.*

As in the case of finite products there are projection maps $p_\beta : \prod_\alpha X_\alpha \to X_\beta$ defined by $p_\beta(x) = x(\beta)$.

Assume now that we have an indexed family of topological spaces $(X_\alpha, \mathbf{T}_\alpha)$. The crucial property of the product topology for $\prod_\alpha X_\alpha$ will be that *it is the unique smallest topology such that every projection map p_β is continuous.*

The preceding condition implies that the product topology should be generated by all sets of the form $p_\beta^{-1}(U_\beta)$ where U_β is open in X_β, and thus the product topology will be arbitrary unions of finite intersections of such sets.

PROPOSITION. *A base for the product topology is given by all open subsets of the form $\prod_\alpha U_\alpha$ where each U_α is open in X_α* **AND** *$U_\alpha = X_\alpha$ for* **all but finitely many** *α.*

Proof. The subsets described in the proposition are finite intersections of sets having the form $p_\beta^{-1}(U_\beta)$; specifically, if Γ is a finite subset of A then the subsets in the proposition have the form

$$\bigcap_{\gamma \in \Gamma} p_\gamma^{-1}(U_\gamma) \ . \blacksquare$$

Another topology on the product is the so-called *box topology* generated by all subsets of the form $\prod_\alpha U_\alpha$ where U_α is an arbitrary open subset of X_α. For finite products these yield the same topology, but this is not true for infinite products. A fairly detailed discussion of the differences appears in Section 19 of Munkres.

Final remark. Theorem 19.6 on page 117 of Munkres gives a fundamentally important property of the product topology (in both the finite and infinite cases).

Finally, here are some facts about finite products that carry over to infinite products. The proofs are essentially the same.

OPENNESS OF PROJECTIONS. *For each β in the indexing set the coordinate projection maps*

$$p_\beta : \prod_\alpha X_\alpha \to X_\beta$$

are open.■

PRODUCTS OF CLOSED SUBSETS. *Let $(X_\alpha, \mathbf{T}_\alpha)$ be a topological space for $\alpha \in A$, and for each α suppose that F_α is a closed subset of X_α. Then $\prod_\alpha F_\alpha$ is a closed subset of $\prod_\alpha X_\alpha$ with respect to the product topology.*■

COROLLARY. *Let $(X_\alpha, \mathbf{T}_\alpha)$ be a topological space for $\alpha \in A$, and for each α suppose that F_α is a closed subset of X_α. Then*

$$\prod_\alpha \overline{A_\alpha} = \overline{\prod_\alpha A_\alpha} \,.■$$

MAPPINGS INTO PRODUCTS. *Let A be a set, and for each $\alpha \in A$ let $f_\alpha : X_\alpha \to Y_\alpha$ be a set-theoretic map. Then there is a unique map*

$$F = \prod_\alpha f_i \ : \ \prod_\alpha X_\alpha \quad \to \quad \prod_\alpha Y_\alpha$$

defined by the conditions

$$\pi_\alpha^Y \circ F = f_\alpha \circ \pi_\alpha^X$$

where π_α^X and π_α^Y denote the $i^{\rm th}$ coordinate projections for $\prod_\alpha X_\alpha$ and $\prod_\alpha Y_\alpha$ respectively. This map is continuous if and only if each f_α is continuous, and it is the identity map if each f_α is an identity map. Finally, if we are also given sets Z_α and (set-theoretic) maps $g_\alpha : Y_\alpha \to Z_\alpha$, and we set G equal to $\prod_\alpha g_\alpha$. Then

$$G \circ F \quad = \quad \prod_\alpha (g_\alpha \circ f_\alpha) \ .$$

Finally we mention one more that is an exercise in Munkres (Theorem 19.4, page 116; see also Exercise 3 on page 118). A proof (probably not the best one) for products of two spaces appears in Section III.1 of these notes.

PRODUCTS AND THE HAUSDORFF PROPERTY. *Let A be a nonempty set, and supposed that X_α is a Hausdorff topological space for each $\alpha \in A$. Then $\prod_\alpha X_\alpha$ is also a Hausdorff space.*■