V. Constructions on Spaces

This material in this unit concerns two basic themes that run throughout geometry — all the way from children's toys to the frontiers of research. One of these is the construction of new objects from old ones by gluing certain subsets together. For example, one can form a circle (actually, a space homeomorphic to a circle) from a closed interval by gluing the two endpoints together. A second theme is the formation of new objects from old ones by first creating several disjoint copies of the original objects and then gluing pieces together in an appropriate fashion. For example, one can form a cube from six pairwise disjoint squares with sides of equal length by gluing the latter together in a suitable way along the edges.

Two widely known examples of such constructions are the Möbius strip and the Klein bottle, and we shall indicate how they can be formed using the ideas presented here. The latter can also be used to show that one can construct a Klein bottle from two Möbius strips by gluing the latter together along their (homeomorphic) edges.

V.I : Quotient spaces

(Munkres, § 22)

We shall adopt a somewhat different approach from the one appearing in Munkres. In mathematics it is often useful to create a quotient object from a mathematical structure and a well-behaved equivalence relation on such an object. For example, if n is a positive integer greater than 1, then one can construct the ring $\mathbf{Z}_n$ of integers modulo n using the equivalence classes of the relation

$$a \equiv b(n) \iff a - b = kn, \text{ some } k \in \mathbf{Z}$$

and the projection from $\mathbf{Z}$ to the set of equivalence classes $\mathbf{Z}_n$ is compatible with the addition and multiplication on both systems. One often says that $\mathbf{Z}_n$ is a *quotient ring* of $\mathbf{Z}$, and further constructions of this sort are indispensable in most of abstract algebra.

Another specific and important example involves the complex numbers, which may be viewed as equivalence classes of real polynomials under the equivalence

$$f(t) \sim g(t) \iff f(t) - g(t) = q(t) \cdot \left(t^2 + 1\right) \quad \text{for some } q(t) \in \mathbf{R}[t]$$

where $\mathbf{R}[t]$ denotes the ring of polynomial forms over the real numbers. As in the case of the integers modulo n, the addition and multiplication maps are given by the sums and products of representatives, and the crucial issue to defining such maps is that if α is equivalent to α' and β is equivalent to β', then $\alpha + \beta$ and $\alpha \cdot \beta$ are equivalent to $\alpha' + \beta'$ and $\alpha' \cdot \beta'$ respectively. Given any other construction for the complex numbers $\mathbf{C}$, there is a unique algebraic isomorphism from the quotient object $\mathbf{R}[t]/\sim$ to $\mathbf{C}$ that sends real numbers to themselves and sends the equivalence class of t to $\sqrt{-1}$.

More generally, quotient constructions arise naturally for many types of mathematical systems, so the following question about topological spaces arises naturally at least from a formal perspective:

Question. *Let X be a topological space and let $\mathcal{R}$ be an equivalence relation on X. Is there a reasonable definition of a topology on the set of equivalence classes $X/\mathcal{R}$?*

One obvious requirement is that the projection map

$$\pi_{\mathcal{R}} : X \to X/\mathcal{R}$$

taking a point x to its $\mathcal{R}$-equivalence class $[x]_{\mathcal{R}}$ (frequently abbreviated to $[x]$) is continuous. One trivial way of achieving this is to take the indiscrete topology on $X/\mathcal{R}$, but something this easy should seem too good to be true (and it is). For example, if X is Hausdorff we would like the topology on $X/\mathcal{R}$ to be Hausdorff, at least as often as possible (there are fundamental examples to show that one cannot always have a Hausdorff topology).

It turns out that the right topology to take for $X/\mathcal{R}$ is the unique maximal topology for which $\pi_{\mathcal{R}}$ is continuous, and it is useful to formulate things somewhat more generally.

Definition. Let $(X, \mathbf{T})$ be a topological space, let Y be a set, and let $f : X \to Y$ be a map of sets. The *quotient topology* $f_*\mathbf{T}$ on Y is defined by the condition

$$V \in f_*\mathbf{T} \text{ if and only if } f^{-1}(V) \in \mathbf{T}.$$

Before proceeding, we need to check that *the construction above yields a topology on* $X/\mathcal{R}$. — The inverse image of the empty set is the empty set and the inverse image of $X/\mathcal{R}$ is X, so $f_*\mathbf{T}$ contains the empty set and $X/\mathcal{R}$. If U_α lies in $X/\mathcal{R}$ for all α, then $f^{-1}(\cup_\alpha U_\alpha) = \cup_\alpha f^{-1}(U_\alpha)$ where each term on the right hand side lies in $\mathbf{T}$ by the definition of $f_*\mathbf{T}$; since the union of open sets in X is again an open subset, it follows that $f^{-1}(\cup_\alpha U_\alpha)$ is open in X which in turn implies that $\cup_\alpha U_\alpha$ belongs to $f_*\mathbf{T}$. Likewise, if U_1 and U_2 are in $f_*\mathbf{T}$, then $f^{-1}(U_1 \cap U_2) = f^{-1}(U_1) \cap f^{-1}(U_2)$, and each term of the right hand side lies in $\mathbf{T}$; since the latter is a topology for x it follows that the right hand side also lies in $\mathbf{T}$, and therefore it follows that $U_1 \cap U_2$ lies in $f_*\mathbf{T}$.

The following are immediate consequences of the definition and the preceding paragraph.

PROPOSITION. *(i) f defines a continuous map from $(X, \mathbf{T})$ to $(Y, f_*\mathbf{T})$,*

(ii) $f_*\mathbf{T}$ contains **every** *topology $\mathbf{U}$ for which $f : (X, \mathbf{T}) \to (Y, \mathbf{U})$ is continuous.*

(iii) A subset $B \subset Y$ is closed with respect to $f_*\mathbf{T}$ if and only if its inverse image $f^{-1}(B)$ is closed with respect to $\mathbf{T}$.

Proof. The first statement follows because a set $V \subset Y$ is open if and only if $f^{-1}(V)$ is open in X. The second is verified by noting that if $\mathbf{U}$ is given as above and $W \in \mathbf{U}$, then $f^{-1}(W) \subset \mathbf{T}$ by continuity, and hence $\mathbf{U} \subset \mathbf{T}$. The third statement holds because B is closed with respect to the quotient topology if and only if $Y - B$ is open, which is true if and only if $f^{-1}(Y - B) = X - f^{-1}(B)$ is open in X, which in turn is true if and only if $f^{-1}(B)$ is closed in X.■

Quotients and morphisms

It is helpful to deal first with some aspects of equivalence class projections that are entirely set-theoretic and relate the discussion here to the approach in Munkres.

If X is a set and $\mathcal{R}$ is an equivalence relation on X, then the equivalence class projection $\pi_{\mathcal{R}}$ is onto. In fact, every onto map can be viewed as an equivalence class projection as follows: If $f : X \to Y$ is an arbitrary onto map of sets, and then one can define an equivalence relation $\mathcal{R}_f$ on X

by $u \sim v$ if and only if $f(u) = f(v)$. There is a canonical 1–1 correspondence $h : Y \to X/\mathcal{R}_f$ which sends the equivalence class $[x]$ to $f(x)$. It is an elementary exercise to show that h is well-defined, 1–1 and onto (verify this!).

The following set-theoretic observation describes an fundamental property of quotient constructions with respect to functions.

PROPOSITION. *Let $f : X \to Y$ be a function, let $\mathcal{R}$ be an equivalence relation on X, and let $p : X \to X/\mathcal{R}$ be the map sending an element to its equivalence class. Suppose that whenever $u \sim_{\mathcal{R}} v$ in X we have $f(u) = f(v)$. Then there is a unique function $g : X/\mathcal{R} \to Y$ such that $g \circ p = f$.*

The point is that a well-defined function is obtained from the formula $g([x]) = f(x)$. Analogous results hold for a wide range of mathematical structures; a version for topological spaces is given immediately after the proof below.

Proof. Suppose that $[x] = [y]$; then by definition of the equivalence relation we have $f(x) = f(y)$, and therefore the formula defining g yields

$$g([x]) = f(x) = f(y) = g([y])$$

which shows that g is indeed well-defined.∎

COROLLARY. *Let f, X, Y, $\mathcal{R}$ be as in the proposition, suppose they satisfy the conditions given there, suppose there are topologies on X and Y such that f is continuous, and put the quotient topology on $X/\mathcal{R}$. Then the unique map g is continuous.*

Suppose that W is open in Y. Then $g^{-1}(W)$ is open in $X/\mathcal{R}$ if and only if $\pi_{\mathcal{R}}^{-1}(g^{-1}(W)$ is open in X. But $g \circ \pi_{\mathcal{R}} = f$, and hence $\pi_{\mathcal{R}}^{-1}(g^{-1}(W) = f^{-1}(W)$. Since f is continuous, the latter is in fact open in X. Therefore $g^{-1}(W)$ is indeed open, so that g is continuous as required.∎

The following relates our approach to that of Munkres.

DEFINITION. Let $f : X \to Y$ be continuous and onto, let $\mathcal{R}_f$ be the equivalence relation described above, and let $h : X/\mathcal{R} \to Y$ be the standard 1–1 onto map described above. By construction, h is continuous if $X/\mathcal{R}$ is given the quotient topology. We say that f *is a quotient map* if h is a homeomorphism.

By the definitions, f is a quotient map if and only if for every $B \subset Y$ we have that B is open in Y if and only if $f^{-1}(B)$ is open in X. — The statement remains true if one replaces "open" by "closed" everywhere.

Remark. In many books the quotient topology is only defined for continuous maps that are onto, so we shall comment on what happens if f is not onto. In this case, if $y \notin f(X)$ then $\{y\}$ is an open and closed subset because $f^{-1}(\{y\}) = \emptyset$ and $\emptyset$ is open and closed in X; more generally, every subset of $Y - f(X)$ is open and closed for the same reason. It also follows that $f(X)$ is open and closed because its inverse image is the open and closed subset X. The quotient topology on the entire space Y is given by the quotient topology on $f(X)$ with respect to the onto map $g : X \to f(X)$ determined by f and the discrete topology on $Y - f(X)$.

Recognizing quotient maps

The following result provides extremely useful criteria for concluding that a continuous onto map is a quotient map.

PROPOSITION. *Suppose that $f : X \to Y$ is continuous and onto, and also assume that f is* **either** *an open mapping or a closed mapping. Then f is a quotient map.*

Proof. We shall only do the case where f is open; the other case follows by replacing "open" with "closed" everywhere in the argument.

We need to show that V is open in Y if and only if $f^{-1}(V)$ is open in X. The ($\Longrightarrow$) implication is true by continuity, and the other implication follows from the hypothesis that f is open because if $f^{-1}(V)$ is open in X then

$$V \;=\; f\big(f^{-1}(V)\big)$$

must be open in Y.■

Exercise 3 on page 145 of Munkres gives an example of a quotient map that is neither open nor closed. We have already given examples of continuous open onto maps that are not closed and continuous closed onto maps that are not open.

COROLLARY. *If X is compact, Y is Hausdorff and $f : X \to Y$ is continuous, then the quotient space $X/\mathcal{R}_f$ is homeomorphic to the subspace $f(X)$ (and hence the quotient is Hausdorff).*

Proof. The preceding arguments yield a continuous 1–1 onto map h from the space $X/\mathcal{R}_f$, which is compact, to the space $f(X)$, which is Hausdorff. Earlier results imply that h is a closed mapping and therefore a homeomorphism.■

Important examples

Throughout this discussion D^n will denote the set of all points x in $\mathbf{R}^n$ satisfying $|x| \leq 1$ (the unit n-disk) and S^{n-1} will denote the subset of all point for which $|x| = 1$ (the unit $(n-1)$-sphere).

We start with the first example at the top of this unit; namely, the circle S^1 is homeomorphic to the quotient of $[0,1]$ modulo the equivalence relation $\mathcal{R}$ whose equivalence classes are the one point sets $\{t\}$ for $t \in (0,1)$ and the two point set $\{0,1\}$. The construction of a homeomorphism is fairly typical; one constructs a continuous onto map from $[0.1]$ to S^1 for which the inverse images of points are the equivalence classes of $\mathcal{R}$. Specifically, let f be the map $[0,1] \to S^1$ defined by $f(t) = \exp(2\pi i t)$.

Non-Hausdorff quotients. We have already mentioned that one can find equivalence relations on Hausdorff spaces for which the quotient spaces are not Hausdorff. The example we shall consider is one with only finitely many equivalence classes: Take the equivalence relation $\mathcal{A}$ on the real line $\mathbf{R}$ whose equivalence classes are all positive real, all negative reals and zero (one verbal description of this relation is that two real numbers are $\mathcal{A}$-related if and only if one is a positive real multiple of another. Then there are three equivalence classes that we shall call $+$, $-$ and 0, and the closed subsets are precisely the following:

$$\emptyset, \quad \mathbf{R}/\mathcal{A}, \quad \{0\}, \quad \{+,\ 0\}, \quad \{-,\ 0\}$$

Since the one point subsets $\{\pm\}$ are not closed in this topology, it is not Hausdorff. Since the quotient topology is the largest topology such that the projection map is continuous, it follows that in this case there is **NO** Hausdorff topology on $\mathbf{R}/\mathcal{A}$ for which the projection

$$\pi_{\mathcal{A}} : \mathbf{R} \to \mathbf{R}/\mathcal{A}$$

is continuous.

Exercise 6 on page 145 of Munkres gives an example of a quotient map on a Hausdorff space where one point subsets in the quotient space are always closed but the quotient is not Hausdorff.

We now come to the constructions of the Möbius strip and Klein bottle. Our description of the former will be designed to reflect the usual construction by gluing together the two short ends of a rectangle whose length is much larger than its width. Let $K > 0$ be a real number, and consider the equivalence relation $\mathcal{M}_K$ on $[-K, K] \times [-1, 1]$ whose equivalence classes are the one point sets $\{\,(s,t)\,\}$ for $|s| < K$ and the two point sets $\{\ (-K, -t).\ (K, t)\ \}$ for each $t \in [-1, 1]$. Mathematically this corresponds to gluing $\{K\} \times [-1, 1]$ to $\{-K\} \times [-1, 1]$ with a twist. A Möbius strip may be viewed as the associated quotient space; note that any two models constructed above are homeomorphic (one can shrink or stretch the first coordinate; details are left to the reader). — Similarly, one can construct a Klein bottle from the cylinder $[-K, K] \times S^1$ by means of the equivalence relation whose equivalence classes are the one point sets $\{\,(s,z)\,\}$ for $|s| < K$ and the two point sets $\{\ (-K, \overline{z}).\ (K, z)\ \}$ for each $z \in S^1$. Mathematically this corresponds to gluing $\{K\} \times S^1$ to $\{-K\} \times S^1$ by the reflection map that interchanges the upper and lower arcs with endpoints ± 1. As in the previous case, the quotient spaces obtained for different values of K are all homeomorphic to each other.

It is physically clear that one can construct the Möbius strip in $\mathbf{R}^3$, and although one cannot find a space homeomorphic to the Klein bottle in $\mathbf{R}^3$ (one needs some algebraic topology to prove this!), some thought strongly suggests that the Klein bottle should be homeomorphic to a subset of $\mathbf{R}^4$ (this has been exploited by numerous science fiction authors). For the sake of completeness (and to prove that the spaces constructed are Hausdorff) we shall prove these realization statements.

The first step is a simple geometric observation:

LEMMA. *For all positive integers p and q the products $S^p \times S^q$ and $S^p \times D^{q+1}$ are homeomorphic to subsets of $\mathbf{R}^{p+q+1}$.*

Proof. We know that $S^p \times \mathbf{R}$ is homeomorphic to the nonzero vectors in $\mathbf{R}^{p+1}$ by the map sending (x, t) to tx because

$$P(v) = \big(|v|^{-1}v, |v|\big)$$

is the inverse. Taking products with $\mathbf{R}^q$ shows that $S^p \times \mathbf{R}^{q+1}$ is homeomorphic to a subset of $\mathbf{R}^{p+q+1}$, and the lemma follows because the former clearly contains $S^p \times S^q$ and $S^p \times D^{q+1}$ ∎

The next step is to observe that one can write the spaces in question as quotients of $[a, b] \times X$ for $a < b \in \mathbf{R}$ and $X = [0, 1]$ or S^1 depending upon whether we are constructing the Möbius strip or Klein bottle; it is only necessary to consider the increasing linear homeomorphism from $[-K, K]$ to $[a, b]$ and substitute a and b for $-K$ and K in the description of the equivalence relations.

The final step is to construct continuous maps from the Möbius strip and Klein bottle to $S^1 \times D^2$ and $S^1 \times S^2$ respectively such that the inverse images of points in the codomains are merely the equivalence classes of the defining relations for the quotient spaces. Since maps from compact spaces into Hausdorff spaces are closed, this means that the images are homeomorphic to the quotient spaces of the domains. For the Möbius strip the map $f : [0,1] \times [-1,1] \to S^1 \times D^2$ is given by

$$f(u,v) = \big(\exp(2\pi i u), v\exp(\pi i u)\big)$$

and for the Klein bottle the map $g : [0,1] \times S^1 \to S^1 \times S^2$ is given by

$$g(u,v) = \big(\exp(2\pi i u), A_u(J(v))\big)$$

where $J : S^1 \to S^2$ is the standard inclusion of $S^1 = S^2 \cap \big(\mathbf{R}^2 \times \{0\}\big)$ and A_u is the orthogonal rotation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \pi u & -\sin \pi u \\ 0 & \sin \pi u & \cos \pi u \end{pmatrix}$$

for which $A_1(J(v)) = J(\overline{v})$. Verifications that these maps have all the desired properties are left to the reader as an exercise.

Final remark. A very extensive treatment of quotient topologies is given in Chapter VI of the text, *Topology*, by J. Dugundji.

V.2 : Sums and cutting and pasting

Most texts and courses on set theory and point set topology do not say much about disjoint union constructions, one reason being that everything is fairly elementary when one finally has the right definitions (two references in print are Sections I.3 and III.4–III.7 of Jänich, *Topology*, and Section 8.7 of Royden, *Real Analysis*). However, these objects arise immediately in a wide range of geometrical and topological constructions of the sort described at the beginning of this unit, including some fundamental examples from later courses in this sequence. A brief but comprehensive treatment seems worthwhile to make everything more precise and to eliminate the need to address the underlying issues in contexts that also involve more sophisticated concepts.

Disjoint union topologies

We have already defined the disjoin union (or set-theoretic sum) of two sets A and B to be the set

$$A \coprod B = (A \times \{1\}) \bigcup (B \times \{2\}) \quad \subset \quad (A \cup B) \times \{1,2\}$$

with injection maps $i_A : A \to A \coprod B$ and $i_B : A \to A \coprod B$ given by $i_A(a) = (a,1)$ and $i_B(b) = (b,2)$. The images of these injections are disjoint copies of A and B, and the union of the images is $A \coprod B$.

Definition. If X and Y are topological spaces, the *disjoint union topology* or *(set-theoretic) sum topology* consists of all subsets having the form $U \coprod V$, where U is open in X and V is open in Y.

We claim that this construction defines a topology on $X \coprod Y$, and the latter is a union of disjoint homeomorphic copies of X and Y such that each of the copies is an open and closed subset. Formally, all this is expressed as follows:

ELEMENTARY PROPERTIES. *The family of subsets described above is a topology for $X \coprod Y$ such that the injection maps i_X and i_Y are homeomorphisms onto their respective images. These images are pairwise disjoint, and they are also open and closed subspaces of $X \coprod Y$. Each injection map is continuous, open and closed.*

Sketch of proof. This is all pretty elementary, but we include it because the properties are so fundamental and the details are not readily available in the standard texts.

Since X and Y are open in themselves and $\emptyset$ is open in both, it follows that $X \coprod Y$ and $\emptyset = \emptyset \coprod \emptyset$ are open in $X \coprod Y$. Given a family of subsets $\{ U_\alpha \coprod V_\alpha \}$ in the so-called disjoint union topology, then the identity

$$\bigcup_\alpha \left(U_\alpha \coprod V_\alpha \right) \quad = \quad \left(\bigcup_\alpha U_\alpha \right) \coprod \left(\bigcup_\alpha V_\alpha \right)$$

shows that the so-called disjoint union topology is indeed closed under unions, and similarly the if $U_1 \coprod V_1$ and $U_2 \coprod V_2$ belong to the so-called disjoint union topology, then the identity

$$\bigcap_{i=1,2} \left(U_i \coprod V_i \right) \quad = \quad \left(\bigcap_{i=1,2} U_i \right) \coprod \left(\bigcap_{i=1,2} V_i \right)$$

shows that the so-called disjoint union topology is also closed under finite intersections. In particular, we are justified in calling this family a topology.

By construction U is open in X if and only if $i_X(U)$ is open in $i_X(X)$, and V is open in Y if and only if $i_Y(V)$ is open in $i_Y(Y)$; these prove the assertions that i_X and i_Y are homeomorphisms onto their images. Since $i_X(X) = X \coprod \emptyset$, it follows that the image of i_X is open, and of course similar considerations apply to the image of i_Y. Also, the identity

$$i_X(X) = \left(X \coprod Y\right) \; - \; i_Y(Y)$$

shows that the image of i_X is closed, and similar considerations apply to the image of i_Y.

The continuity of i_X follows because every open set in $X \coprod Y$ has the form $U \coprod V$ where U and V are open in X and Y respectively and

$$i_X^{-1}\left(U \coprod V\right) \quad = \quad U$$

with similar conditions valid for i_Y. The openness of i_X follows immediately from the identity $i_X(U) = U \coprod \emptyset$ and again similar considerations apply to i_Y. Finally, to prove that i_X is closed, let $F \subset X$ be closed. Then $X - F$ is open in X and the identity

$$i_X(F) = F \coprod \emptyset = \left(X \coprod Y\right) - \left((X - F) \coprod Y\right)$$

shows that $i_X(F)$ is closed in $X \coprod Y$; once more, similar considerations apply to i_Y.■

IMMEDIATE CONSEQUENCE. *The closed subsets of $X \coprod Y$ with the disjoint union topology are the sets of the form $E \coprod F$ where E and F are closed in X and Y respectively.*■

If the topologies on X and Y are clear from the context, we shall generally assume that the $X \coprod Y$ is furnished with the disjoint union topology unless there is an explicit statement to the contrary.

Since the disjoint union topology is not covered in many texts, we shall go into more detail than usual in describing their elementary properties.

FURTHER ELEMENTARY PROPERTIES. (*i*) *If X and Y are discrete, then so is $X \coprod Y$.*

(*ii*) *If X and Y are Hausdorff, then so is $X \coprod Y$.*

(*iii*) *If X and Y are homeomorphic to metric spaces, then so is $X \coprod Y$.*

(*iv*) *If $f : X \to W$ and $g : Y \to W$ are continuous maps into some space W, then there is a unique continuous map $h : X \coprod Y \to W$ such that $h \circ i_X = f$ and $h \circ i_Y = g$.*

(*v*) *The spaces $X \coprod Y$ and $Y \coprod X$ are homeomorphic for all X and Y. Furthermore, if Z is a third topological space then there is an "associativity" homeomorphism*

$$\left(X \coprod Y\right) \coprod Z \cong X \coprod \left(Y \coprod Z\right)$$

(in other words, the disjoint sum construction is commutative and associative up to homeomorphism).

Sketches of proofs. (*i*) A space is discrete if every subset is open. Suppose that $E \subset X \coprod Y$. Then E may be written as $A \coprod B$ where $A \subset X$ and $B \subset Y$. Since X and Y are discrete it follows that A and B are open in X and Y respectively, and therefore $E = A \coprod B$ is open in $X \coprod Y$. Since E was arbitrary, this means that the disjoint union is discrete.

(*ii*) If one of the points p, q lies in the image of X and the other lies in the image of Y, then the images of X and Y are disjoint open subsets containing p and q respectively. On the other hand, if both lie in either X or Y, let V and W be disjoint open subsets containing the preimages of p and q in X or Y. Then the images of V and W in $X \coprod Y$ are disjoint open subsets that contain p and q respectively.

(*iii*) As noted in Theorem 20.1 on page 121 of Munkres, if the topologies on X and Y come from metrics, one can choose the metrics so that the distances between two points are ≤ 1. Let $\mathbf{d}_X$ and $\mathbf{d}_Y$ be metrics of this type.

Define a metric $\mathbf{d}^*$ on $X \coprod Y$ by $\mathbf{d}_X$ or $\mathbf{d}_Y$ for ordered pairs of points (p, q) such that both lie in the image of i_X or i_Y respectively, and set $\mathbf{d}^*(p, q) = 2$ if one of p, q lies in the image of i_X and the other lines in the image of i_Y. It follows immediately that $\mathbf{d}^*$ is nonnegative, is zero if and only if $p = q$ and is symmetric in p and q. All that remains to check is the Triangle Inequality:

$$\mathbf{d}^*(p, r) \leq \mathbf{d}^*(p, q) + \mathbf{d}^*(q, r)$$

The verification breaks down into cases depending upon which points lie in the image of one injection and which lie in the image of another. If all three of p, q, r lie in the image of one of the injection maps, then the Triangle Inequality for these three points is an immediate consequence of the corresponding properties for $\mathbf{d}_X$ and $\mathbf{d}_Y$. Suppose now that p and r lie in the image of one injection and q lies in the image of the other. Then we have $\mathbf{d}^*(p, r) \leq 1$ and

$$\mathbf{d}^*(p, q) + \mathbf{d}^*(q, r) = 2 + 2 = 4$$

so the Triangle Inequality holds in these cases too. Finally, if p and r lie in the images of different injections, then either p and q lie in the images of different injections or else q and r lie in the images of different injections. This means that $\mathbf{d}^*(p, r) = 2$ and $\mathbf{d}^*(p, q) + \mathbf{d}^*(q, r) \geq 2$, and consequently the Triangle Inequality holds for **all** ordered pairs (p, r).

(*iv*) Define $h(x, 1) = f(x)$ and $h(y, 2) = g(y)$ for all $x \in X$ and $y \in Y$. By construction $h \circ i_X = f$ and $h \circ i_Y = g$, so it remains to show that h is continuous and there is no other continuous map satisfying the functional equations. The latter is true for set theoretic reasons; the equations specify the behavior of h on the union of the images of the injections, but this image is the entire disjoint union. To see that h is continuous, let U be an open subset of X, and consider the inverse image $U^* = h^{-1}(U)$ in $X \coprod Y$. This subset has the form $U^* = V \coprod W$ for some subsets $V \subset X$ and $W \subset Y$. But by construction we have

$$V = i_X^{-1}(U^*) = i_X^{-1} \circ h^{-1}(U) = f^{-1}(U)$$

and the set on the right is open because f is continuous. Similarly,

$$W = i_Y^{-1}(U^*) = i_Y^{-1} \circ h^{-1}(U) = g^{-1}(U)$$

so that the set on the right is also open. Therefore $U^* = V \coprod W$ where V and W are open in X and Y respectively, and therefore U^* is open in $X \coprod Y$, which is exactly what we needed to prove the continuity of h.

(*v*) We shall merely indicate the main steps in proving these assertions and leave the details to the reader as an exercise. The homeomorphism τ from $X \coprod Y$ to $Y \coprod X$ is given by sending $(x, 1)$

to $(x,2)$ and $(y,2)$ to $(y,1)$; one needs to check this map is 1–1, onto, continuous and open (in fact, if τ_{XY} is the map described above, then its inverse is τ_{YX}). The "associativity homeomorphism" sends $((x,1),1)$ to $(x,1)$, $((y,2),1)$ to $((y,1),2)$, and $(z,2)$ to $((z,2),2)$. Once again, one needs to check this map is 1–1, onto, continuous and open.■

COMPLEMENT. *There is an analog of Property (iv) for untopologized sets.*

Perhaps the fastest way to see this is to make the sets into topological spaces with the discrete topologies and then to apply (*i*) and (*iv*).■

Property (*iv*) is dual to a fundamental property of product spaces. Specifically, oredered pairs of maps from a fixed object A to objects B and C correspond to maps from A into $B \times C$, while oredered pairs of maps going **TO** a fixed object A and coming **FROM** objects B and C correspond to maps from $B \coprod C$ into A. For this reason one often refers to $B \coprod C$ as the *coproduct* of B and C (either as sets or as topological spaces); this is also the reason for denoting disjoint unions by the symbol $\coprod$, which is merely the product symbol $\prod$ turned upside down.

Copy, cut and paste constructions

Frequently the construction of spaces out of pieces proceeds by a series of steps where one takes two spaces, say A and B, makes disjoint copies of them, finds closed subspaces C and D that are homeomorphic by some homeomorphism h, and finally glues A and B together using this homeomorphism. For example, one can think of a rectangle as being formed from two right triangles by gluing the latter along the hypotenuse. Of course, there are also many more complicated examples of this sort.

Formally speaking, we can try to model this process by forming the disjoint union $A \coprod B$ and then factoring out by the equivalence relation

$$x \sim y \iff x = y \quad \text{or}$$

$$x = i_A(a),\ y = i_B(h(a)) \quad \text{for some } a \in A \quad \text{or}$$

$$y = i_A(a),\ x = i_B(h(a)) \quad \text{for some } a \in A.$$

It is an elementary but tedious exercise in bookkeeping to to verify that this defines an equivalence relation (the details are left to the reader!). The resulting quotient space will be denoted by

$$A \bigcup_{h:C \equiv D} B.$$

As a test of how well this approach works, consider the following question:

Scissors and Paste Problem. *Suppose we are given a topological space X and closed subspaces A and B such that $X = A \cup B$. If we take $C = D = A \cap B$ and let h be the identity homeomorphism, does this construction yield the original space X?*

One would expect that the answer is yes, and here is the proof:

Retrieving the original space. Let Y be the quotient space of $A \coprod B$ with respect to the equivalence relation, and let $p : A \coprod B \to Y$ be the quotient map. By the preceding observations, there is a unique continuous map $f : A \coprod B \to X$ such that $f \circ i_A$ and $f \circ i_B$ are the inclusions

$A \subset X$ and $B \subset X$ respectively. By construction, if $u \sim v$ with respect to the equivalence relation described above, then $f(u) = f(v)$, and therefore there is a unique continuous map $h : Y \to X$ such that $f = h \circ p$. We claim that h is a homeomorphism. First of all, h is onto because the identities $h \circ p \circ i_A = \text{inclusion}_A$ and $h \circ p \circ i_B = \text{inclusion}_B$ imply that the image contains $A \cup B$, which is all of X. Next, h is 1–1. Suppose that $h(u) = h(v)$ but $u \neq v$, and write $u = p(u')$, $v = p(v')$. The preceding identities imply that h is 1-1 on both A and B, and therefore one of u', v' must lie in A and the other in B. By construction, it follows that the inclusion maps send u' and v' to the same point in X. But this means that u' and v' correspond to the same point in $A \cap B$ so that $u = p(u') = p(v') = v$. Therefore the map h is 1–1. To prove that h is a homeomorphism, it suffices to show that h takes closed subsets to closed subsets. Let F be a closed subset of Y. Then the inverse image $p^{-1}(F)$ is closed in $A \coprod B$. However, if we write write $h(F) \cap A = P$ and $h(f) \cap B = Q$, then it follows that $p^{-1}(F) = i_A(P) \cup i_B(Q)$. Thus $i_A(P) = p^{-1}(F) \cap i_A(A)$ and $i_B(Q) = p^{-1}(F) \cap i_B(B)$, and consequently the subsets $i_A(P)$ and $i_B(Q)$ are closed in $A \coprod B$. But this means that P and Q are closed in A and B respectively, so that $P \cup Q$ is closed in X. Therefore it suffices to verify that $h(F) = P \cup Q$. But if $x \in F$, then the surjectivity of p implies that $x = p(y)$ for some $y \in p^{-1}(F) = i_A(P) \cup i_B(Q)$; if $y \in i_A(P)$ then we have

$$h(x) = h(p(y)) = f(y) = f \circ i_A(y) = y$$

for some $y \in P$, while if $y \in i_B(Q)$ the same sorts of considerations show that $h(x) = y$ for some $y \in Q$. Hence $h(F)$ is contained in $P \cup Q$. On the other hand, if $y \in P$ or $y \in Q$ then the preceding equations for P and their analogs for Q show that $y = h(p(y))$ and $p(y) \in F$ for $y \in P \cup Q$, so that $P \cup Q$ is contained in $h(F)$ as required.■

One can formulate an analog of the scissors and paste problem if A and B are open rather than closed subset of X, and once again the answer is that one does retrieve the original space. The argument is similar to the closed case and is left to the reader as an exercise.

Examples. Many examples for the scissors and paste theorem can be created involving subsets of Euclidean 3-space. For example, as noted before one can view the surface of a cube as being constructed by a sequence of such operations in which one adds a solid square homeomorphic to $[0,1]^2$ to the space constructed at the previous step. Our focus here will involve examples of objects in 4-dimensional space that can be constructed by a single scissors and paste construction involving objects in 3-dimensional space.

1. The *hypersphere* $S^3 \subset \mathbf{R}^4$ is the set of all points (x, y, z, w) whose coordinates satisfy the equation

$$x^2 + y^2 + z^2 + w^2 \quad = \quad 1$$

and it can be constructed from two 3-dimensional disks by gluing them together along the boundary spheres. An explicit homeomorphism

$$D^3 \bigcup_{\mathrm{id}(S^2)} D^3 \longrightarrow S^3$$

can be constructed using the maps

$$f_{\pm}(x, y, z) = \left(x, y, z, \sqrt{1 - x^2 - y^2 - z^2}\right)$$

on the two copies of D^3. The resulting map is well defined because the restrictions of $f_{\pm}$ to S^2 are equal.

2. We shall also show that the Klein bottle can be constructed by gluing together two Möbius strips along the simple closed curves on their edges. Let $g_\pm : [-1,1] \to S^1$ be the continuous 1–1 map sending t to $\left(\pm\sqrt{1-t^2}, t\right)$. It then follows that the images $F_\pm$ of the maps $\mathrm{id}_{[0,1]} \times [-1,1]$ satisfy $F + \cup F_- = [0,1] \times S^1$ and $F_+ \cap [0,1] \times \{-1,1\}$. If $\varphi : [0,1] \times S^1 \to \mathbf{K}$ is the quotient projection to the Klein bottle, then it is relatively elementary to verify that each of the sets $\varphi(F_\pm)$ is homeomorphic to the Möbius strip (look at the equivalence relation given by identifying two points if they have the same images under $\varphi \circ g_\pm$) and the intersection turns out to be the set $\varphi(F_+) \cap \varphi(F_-)$, which is homeomorphic to the edge curve for either of these M̈bius strips.

Disjoint unions of families of sets

As in the case of products, one can form disjoint unions of arbitrary finite collections of sets or spaces recursively using the construction for a pair of sets. However, there are also cases where one wants to form disjoint unions of infinite collections, so we shall sketch how this can be done, leaving the proofs to the reader as exercises.

Definition. If A is a set and $\{ X_\alpha \mid \alpha \in A \}$ is a family of sets indexed by A, the *disjoint union* (or set-theoretic sum)

$$\coprod_{\alpha \in A} X_\alpha$$

is the subset of all

$$(x, \alpha) \in \left(\bigcup_{\alpha \in S} X_\alpha \right) \times A$$

such that $x \in X_\alpha$.

This is a direct generalization of the preceding construction, which may be viewed as the special case where $A = \{1, 2\}$. For each $\beta \in A$ one has an injection map

$$i_\beta : X_\beta \to \coprod_{\alpha \in A} X_\alpha$$

sending x to (x, β); as before, the images of i_β and i_γ are disjoint if $\beta \neq \gamma$ and the union of the images of the maps i_α is all of $\coprod_\alpha X_\alpha$.

Notation. In the setting above, suppose that each X_α is a topological space with topology $\mathbf{T}_\alpha$. Let $\sum_\alpha \mathbf{T}_\alpha$ be the set of all disjoint unions $\coprod_\alpha U_\alpha$ where U_α is open in X_α for each α.

As in the previous discussion, this defines a topology on $\coprod_\alpha X_\alpha$, and the basic properties can be listed as follows:

[1] *The family of subsets $\sum_\alpha \mathbf{T}_\alpha$ defines a topology for $\coprod_\alpha X_\alpha$ such that the injection maps i_α are homeomorphisms onto their respective images. the latter are open and closed subspaces of $\coprod_\alpha X_\alpha$, and each injection is continuous, open and closed.*

[2] *The closed subsets of $\coprod X_\alpha$ with the disjoint union topology are the sets of the form $\coprod F_\alpha$ where F_α is closed in X_α for each α.*

[3] *If each X_α is discrete then so is $\coprod_\alpha X_\alpha$.*

[4] *If each X_α is Hausdorff then so is $\coprod_\alpha X_\alpha$.*

[5] *If each X_α is homeomorphic to a metric space, then so is $\coprod_\alpha X_\alpha$.*

[6] *If for each α we are given a continuous function $f : X_\alpha \to W$ into some fixed space W, then there is a unique continuous map $h : \coprod_\alpha X_\alpha \to W$ such that $h \circ i_\alpha = f_\alpha$ for all α.*

The verifications of these properties are direct extensions of the earlier arguments, and the details are left to the reader.∎

In linear algebra one frequently encounters vector spaces that are isomorphic to direct sums of other spaces but not explicitly presented in this way, and it is important to have simple criteria for recognizing situations of this type. Similarly, in working with topological spaces one frequently encounters spaces that are homeomorphic to disjoint unions but not presented in this way, and in this context it is also convenient to have a simple criterion for recognizing such objects.

RECOGNITION PRINCIPLE. *Suppose that a space Y is a union of pairwise disjoint subspaces X_α, each of which is open and closed in Y. Then Y is homeomorphic to $\coprod_\alpha X_\alpha$.*

Proof. For each $\alpha \in A$ let $j_\alpha : X_\alpha \to Y$ be the inclusion map. By [6] above there is a unique continuous function

$$J : \coprod_\alpha X_\alpha \to Y$$

such that $J \circ i_\alpha = j_\alpha$ for all α. We claim that J is a homeomorphism; in other words, we need to show that J is 1–1 onto and open. Suppose that we have $(x_\alpha, \alpha) \in i_\alpha(X_\alpha)$ and $(z_\beta, \beta) \in i_\beta(X_\beta)$ such that $J(x_\alpha, \alpha) = J(z_\beta, \beta)$. By the definition of J this implies $i_\alpha(x_\alpha) = i_\beta(z_\beta)$. Since the images of i_α and i_β are pairwise disjoint, this means that $\alpha = \beta$. Since i_α is an inclusion map, it is 1–1, and therefore we have $x_\alpha = z_\beta$. The proof that J is onto drops out of the identities

$$J\left(\coprod_\alpha X_\alpha\right) = J\left(\bigcup_\alpha i_\alpha(X_\alpha)\right) = \bigcup_\alpha J\left(i_\alpha(X_\alpha)\right) = \bigcup_\alpha j_\alpha(X_\alpha) = Y\ .$$

Finally, to prove that J is open let W be open in the disjoint union, so that we have

$$W = \coprod_\alpha U_\alpha$$

where each U_α is open in the corresponding X_α. It then follows that $J(W) = \cup_\alpha U_\alpha$. But for each α we know that U_α is open in X_α and the latter is open in Y, so it follows that each U_α is open in Y and hence that $J(W)$ is open.∎