VI. Spaces with additional properties

This unit is essentially a continuation of Unit III, and it deals with two main issues:

(1) Topological properties of spaces that are not compact but still have some important properties in common with compact spaces (open subsets of $\mathbf{R}^n$ are particularly important examples in this connection).

(2) Recognition of topological spaces that come from metric spaces. We shall concentrate on questions involving spaces constructed from reasonable pieces and merely state the general results with references to Munkres for the proofs.

In a ten week beginning graduate course it is not possible to cover everything about topological spaces that is useful in a broad range of mathematical contexts and/or for further courses in geometry and topology. Two particularly worthwhile topics of this sort are paracompact spaces (Munkres, § 41) and the compact-open topologies on spaces of continuous functions (Munkres, § 46). Another interesting topic, known as dimension theory (Munkres, § 50), deals with the following natural question: *How can one use topology to define the dimension of a topological space (as an integer* ≥ -1 *or* ∞*) such that the topological dimension of* $\mathbf{R}^n$ *is precisely* n*?* It seems reasonable to expect that $\mathbf{R}^m$ and $\mathbf{R}^n$ are not homeomorphic if $m \neq n$, and such a definition would yield this as a simple corollary. Proper mappings are another important topic that definitely would be worth discussing; these mappings are discussed at a number of points in Munkres (where they are called *perfect* maps), and the files `proper.*` in the course directory contain further information.

An extremely comprehensive listing of properties of topological spaces, along with theorems and examples to describe the logical interrelationships between the concepts, is contained in the book, *Counterexamples in Topology*, by L. A. Steen and J. A. Seebach; the reference charts at the end are particularly helpful for obtaining a good overview of this area. Another book containing a very substantial amount of information on different properties of topological spaces is the text, *Topology*, by J. Dugundji.

In analysis one considers a large variety of topological vector spaces (each one point subset is closed, and both addition and scalar multiplication are continuous), and questions about the metrizability of these spaces (and the properties of such metrics) arise naturally. This topic is discussed specifically in Chapter 1 of Rudin, *Functional Analysis*.

VI.1 : Second countable spaces

(Munkres, § 30)

We have already noted that continuous functions from the unit interval to a Hausdorff space are completely determined by their restrictions to the rational points of the interval. In fact, this property holds for **all** subsets of Euclidean spaces. The proof of this depends upon the existence of a countable dense subset and the fact that the topology comes from a metric. There are two useful equivalent characterizations of such metric spaces, and one of the conditions implies the other two for arbitrary topological spaces.

Definitions. If X is a topological space then X is said to be

- (i) *separable* if it has a countable dense subset,
- (ii) *second countable* (or to satisfy the second countability axiom) if there is a countable base for the topology (*i.e.*, a countable family $\mathcal{B}$ of open sets such that every open set is a union of sets in $\mathcal{B}$,
- (iii) *Lindelöf* (or to have the Lindelöf property) if every open covering has a countable subcovering.

The logical relations between these concepts are given as follows:

THEOREM. *A topological space that is second countable is also separable and Lindelöf, and a metric space that is either separable or Lindelöf is also second countable.*

In particular, all three concepts are equivalent for metric spaces. Examples exist to show that separable or Lindelöf spaces need not be second countable, and for arbitrary topological spaces there is no relation between separability and the Lindelöf property (there are examples where each is true and the other is false).

The book, *Counterexamples in Topology*, by Steen and Seebach, is a standard reference for examples of topological spaces which have one property but not another.

Implications of second countability

We shall begin by showing that second countability implies the other properties.

Separability. Let $\mathcal{B} = \{U_1, U_2, \cdots\}$ be a countable base for the topology, and form the countable subset $A \subset X$ by picking a point $a_i \in U_i$ for each i. To show that A is dense in X we need to show that every open subset of X contains a point in A. Since every open set $W \subset X$ is a union of sets in $\mathcal{B}$ there is at least one U_j that is contained in W. We then have $a_j \in U_j \subset W$. Therefore $\overline{A} = X$.

Lindelöf property. Once again let $\mathcal{B} = \{U_1, U_2, \cdots\}$ be a countable base for the topology. Given an open covering $\mathcal{U} = \{\mathcal{W}_\alpha\}$ of X, let

$$\mathcal{B}_0 = \{V_1, V_2, \cdots\}$$

be the (countable) family of all basic open sets that are contained in some element of $\mathcal{W}$. It follows that $\mathcal{B}_0$ is an open covering of X (because X is a union of the W_α and each W_α is a union of sets in $\mathcal{B}_0$). If for each j we pick $\alpha(j)$ such that $V_j \subset W_{\alpha(j)}$ it follows that

$$\mathcal{W}_O = \{W_{\alpha(1)}, W_{\alpha(2)}, \cdots\}$$

is a countable subcovering of $\mathcal{W}$.■

For the time being we shall simply note that $\mathbf{R}^n$ is an example of a separable metric space; a countable dense subset is given by the subset $\mathbf{Q}^n$ of points whose coordinates are all rational numbers. The results below will show that $\mathbf{R}^n$ and its subspaces also have the other two properties.

Second countability behaves well with respect to some standard operations on topological spaces:

PROPOSITION. *A subspace of a second countable space is second countable, and the product of two (hence finitely many) second countable spaces is second countable.*

This is stated and proved as Theorem 30.2 on page 191 of Munkres.

The reverse implications for metric spaces

We shall show that each of the other two properties implies second countability for metric spaces. The first of these implications will prove that $\mathbf{R}^n$ is a separable metric space.

Separable metric spaces are second countable. Let $A = \{a_1, a_2, \cdots\}$ be a countable dense subset and consider the countable family of open sets $W_{m,n} = N_{1/m}(a_n)$. Given an open set U in X and a point $p \in X$, let $\varepsilon > 0$ be chosen so that $N_\varepsilon(x) \subset U$. Choose m and n such that $1/2m < \varepsilon$ and $\mathbf{d}(x, a_n) < 1/2m$. If we set $W(x) = W_{m,n}$ it then follows that $x \in W(x) \subset U$, and consequently we also have $U = \cup_x W(x)$, which shows that the countable family $\mathcal{W} = \{ W_{m,n} \}$ is a base for the topology.■

COROLLARY. *The space $\mathbf{R}^n$ is a second countable space.*■

The preceding implications have the following useful consequence.

PROPOSITION. *If X is a separable metric space and $A \subset X$, then A is also separable.*■

This follows because separable metric implies second countable and the latter implies separable. One can construct examples of separable topological spaces that have nonseparable subspaces.

COROLLARY. *Every subset S of $\mathbf{R}^n$ has a countable dense subset.*■

Note that the subset S might not contain any points at all from some arbitrary countable dense subset $D \subset \mathbf{R}^n$.

Proof that Lindelöf metric spaces are second countable. For each positive integer n let $\mathcal{U}_n$ be the family of all sets $N_{1/n}(x)$ where x runs through all the points of X. Then $\mathcal{U}_n$ is an open covering of X and consequently has a countable subcovering $\mathcal{W}_n$. We claim that $\mathcal{W} = \cup_n \mathcal{W}_n$ is a base for the metric topology.

Let V be an open subset of X and let $x \in V$. Then there is some positive integer M such that $N_{1/n}(x) \subset V$ for all $n > M$. Choose an open set $N'(x)$ from $\mathcal{W}_{2n}$ such that $x \in N'$. It then follows that $N'(x) \subset N_{1/n}(x) \subset V$, and therefore we have $\cup_x N'(x) = V$, which shows that $\mathcal{W}$ is a countable base for the metric topology on X.

A compact space is automatically Lindelöf, and therefore we have the following consequence for compact metric spaces:

PROPOSITION. *A compact metric space is second countable and has a countable dense subset.*■

VI.2 : Compact spaces – II

(Munkres, §§ 26, 27, 28)

This unit contains some additional results on compact spaces that are useful in many contexts.

Sequential compactness for metric spaces

We have shown that infinite sequences in compact metric spaces always have convergent subsequences and noted that the converse is also true. Here is a proof of that converse:

THEOREM. *If X is a metric space such that every infinite sequence has a convergent subsequence, then X is compact.*

Proof. The idea is to show first that X is separable, then to use the results on second countability to show that X is Lindelöf, and finally to extract a finite subcovering from a countable subcovering.

Proof that a metric space is separable if every infinite sequence has a convergent subsequence. Let $\varepsilon > 0$ be given. We claim that there is a finite collection of points $Y(\varepsilon)$ such that the finite family

$$\{ N_\varepsilon(y) \mid y \in Y(\varepsilon) \}$$

is an open covering of X. — Suppose that no such finite set exists. Then one can recursively construct a sequence $\{ x_n \}$ in X such that

$$N_\varepsilon(x_n) \ \cap \ \Big(\bigcup_{i<n} N_\varepsilon(x_i) \Big) = \emptyset \ .$$

By construction we have that $\mathbf{d}(x_p, x_q) \geq \varepsilon$ if $p \neq q$, and therefore $\{ x_n \}$ has no Cauchy (hence no convergent) subsequence.

If $A = \cup_n Y(1/n)$ then A is countable and for every $\delta > 0$ and $x \in X$ there is a point of A whose distance to x is less than δ, and therefore A is dense in X.

Proof that open coverings have finite subcoverings. Since X is separable and metric, it is second countable, and therefore it also has the Lindelöf property. Therefore given an open covering $\mathcal{W}$ of X we can fine a countable subcovering $\mathcal{U} = \{U_1, U_2, \ \cdots \}$. We need to extract a finite subcovering from $\mathcal{U}$.

For each positive integer n let

$$E_n = X - \Big(\bigcup_{i \leq n} U_n \Big) \ .$$

Then each E_n is closed and $E_n \supset E_{n+1}$ for all n. If some E_n is empty then the first n sets in $\mathcal{U}$ form a finite subcovering; in fact, if some E_n is finite, then there still is a finite subcovering (take the first n sets in $\mathcal{U}$ and add one more set from $\mathcal{U}$ for each point in the intersection). Therefore the proof reduces to finding a contradiction if one assumes that each E_n is infinite.

If each E_n is infinite, then one can find a sequence of *distinct* points y_n such that $y_j \in E_j$ for each j. The assumption on X implies that the infinite sequence $\{ y_j \}$ has a convergent subsequence, say $\{ y_{k(j)} \}$. Let y^* be the limit of this subsequence. By construction, $y_m \in E_n$ if $m \geq n$, and because each E_i is closed it follows that $y^* \in E_{k(j)}$ for all j. Since the sequence of closed sets $\{ E_n \}$ is decreasing, it follows that

$$\bigcap_n E_n \quad = \quad \bigcap_j E_{k(j)}$$

and that y^* belongs to this intersection. But by construction the set $\cap_n E_n$ is empty because $\mathcal{U}$ is a countable open covering for X, and therefore we have a contradiction; it follows that $\mathcal{U}$ must have a finite subcovering, and therefore X must be compact.∎

Wallace's Theorem

The following result is related to the proof that a product of two compact spaces is compact, and it turns out to be extremely useful in many contexts.

WALLACE'S THEOREM. *Let X and Y be topological spaces, let A and B be compact subsets of X and Y, and let W be an open subset of $X \times Y$ that contains A and B. Then there are open subsets U and V of X and Y respectively such that $A \subset U$, $B \subset V$ and*

$$A \times B \ \subset \ U \times V \ \subset \ W \ .$$

Proof. Given $p = (x, y) \in A \times B$ one can find open sets U_p and V_p in X and Y respectively such that $x \in U_p$, $y \in V_p$ and $U_p \times V_p \subset W$.

For a fixed $b \in B$ the open sets $U_p \times V_p$ define an open covering of the compact subset $A \times \{b\}$ and hence there is a finite subcovering associated to the family of sets

$$\mathcal{A}_b = \{U_{p(1)} \times V_{p(1)}, \ \cdots \ , U_{p(M(b))} \times V_{p(M(b))}\} \ .$$

If we take $V_b^{\#}$ to be the intersection of the sets $V_{p(j)}$ and $U_b^{\#}$ to be the union of the sets $U_{p(j)}$, it follows that

$$A \times \{b\} \ \subset \ U_b^{\#} \times V_b^{\#} \ \subset \ W$$

for each $b \in B$

The family of open subsets $\{ V_b^{\#} \}$ in Y defines an open covering of the compact subspace b, and therefore there is a finite subcovering associated to some family of sets

$$\mathcal{F} = \{U_{b(1)}^{\#} \times V_{b(1)}^{\#}, \ \cdots \ , U_{b(N)}^{\#} \times V_{b(N)}^{\#}\} \ .$$

If we now take U to be the intersection of the sets $U_{b(i)}^{\#}$ and V to be the union of the sets $V_{b(i)}^{\#}$, it follows that

$$A \times B \ \subset \ U \times V \ \subset \ W$$

as required.∎

VI.3 : Separation axioms

(Munkres, §§ 31, 32, 33, 35)

In some sense the definitions for topological spaces and metric spaces present an interesting contrast. While it is clear that one does not need the full force of the properties of a metric space to prove many basic results in point set theory, it is also apparent that one needs something more than the austere structure of a topological space to go beyond a certain point. Up to this point we have introduced several conditions like the Hausdorff Separation Property which suffice for proving a number of basic results. This property is just one of a list of increasingly stronger properties that lie between a topological space with no further conditions at all and a topological space that comes from a metric space. There are many important examples of topological spaces in topology, geometry and analysis that are not homeomorphic to metric spaces (*e.g.*, many infinite-dimensional objects in algebraic topology and the so-called weak topologies on Banach spaces), and such objects are one motivation for introducing concepts somewhere between metric spaces and arbitrary topological spaces. A second motivation, which can be viewed as interesting for its own sake as well as its usefulness in certain situations, is the following:

METRIZABILITY PROBLEM. *What sorts of topological conditions are necessary or sufficient for a topological space to be homeomorphic to a metric space?*

We shall consider this problem in some detail as the final topic of the course.

The $\mathbf{T_i}$ *conditions*

The traditional way of organizing the separation properties that may or may not hold in a topological space involves a list of statements $\mathbf{T_i}$ where the subscript is some rational number and the strength of the condition increases with the index (so if $\mathbf{i} > \mathbf{j}$ then $\mathbf{T_i} \Longrightarrow \mathbf{T_j}$). We shall only deal with the statements for $\mathbf{i} = 0,\ 1,\ 2,\ 3,\ 3\frac{1}{2}$ and 4. Definitions for certain other values of i (and a great deal more) may be found online in the web sites, the paper and the reference to Munkres listed below:

> `www.wikipedia.org/wiki/Separation_axiom` — This list is pretty comprehensive.
>
> `at.yorku.ca/i/d/e/b75.htm` — This points to electronic copies of a paper, *"Definition bank" in general topology*, by G. V. Nagalagi, and one has many choices of format for downloading this paper. The listing of properties $\mathbf{T_i}$ for $\mathbf{i} < 1$ is particularly extensive.
>
> The paper, *Espaces* $\mathbf{T}_{1\frac{1}{2}}$, by Carlos A. Infantozzi [Proceedings of the International Symposium on Topology and its Applications (Budva, 1972), pp. 116–122; Savez Društava Mat. Fiz. i Astronom., Belgrade, 1973], deals with the case $\mathbf{i} = 1\frac{1}{2}$.
>
> Exercise 6(b) on page 213 of Munkres describes a natural candidate for $\mathbf{T_6}$.

To answer obvious questions about the choice of terminology, the symbolism $\mathbf{T_i}$ comes from the German word *Trennung*, which means separation (the terms were introduced in the classic book of Alexandroff and Hopf on topology, which was published in the nineteen thirties and written in German). In any case, here are the most important of the separation properties for this course:

Definitions. **(T0)** A topological space X is said to be a $\mathbf{T_0}$ space if for each pair of points $x,\, y \in X$ there is an open set containing one but not the other.

(T1) A topological space X is said to be a $\mathbf{T_1}$ space if for each $x \in X$ the one point set $\{x\}$ is closed in X.

(T2) A topological space X is said to be a $\mathbf{T_2}$ space if it has the Hausdorff Separation Property.

(T3) A topological space X is said to be a $\mathbf{T_3}$ space if it is a $\mathbf{T_1}$ space and is also *regular*: Given a point $x \in X$ and an open set U containing x, there is an open subset V such that

$$x \in V \subset \overline{V} \subset U$$

or equivalently that if $x \in X$ and F is a closed subset not containing X, then there are disjoint open subsets V and W such that $x \in V$ and $F \subset W$.

(T3$\frac{1}{2}$) A topological space X is said to be a $\mathbf{T_{3\frac{1}{2}}}$ space if it is a $\mathbf{T_1}$ space and is also *completely regular*: Given a point $x \in X$ and a closed subset $F \subset X$ not containing x, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f = 1$ on F.

(T4) A topological space X is said to be a $\mathbf{T_4}$ space if it is a $\mathbf{T_1}$ space and is also *normal*: Given a closed subset $E \subset X$ and an open set U containing x, there is an open subset V such that

$$E \subset V \subset \overline{V} \subset U$$

or equivalently that if E and F are disjoint closed subsets of X, then there are disjoint open subsets V and W such that $E \subset V$ and $F \subset W$.

Most of the implications

$$\text{if } \mathbf{i} > \mathbf{j}\text{ , then } \mathbf{T_i} \Longrightarrow \mathbf{T_j}$$

are clear or have already been established, the main exceptions involving the case $\mathbf{i} = 3\frac{1}{2}$. To see that $\mathbf{T_{3\frac{1}{2}}}$ implies $\mathbf{T_3}$, given x and F let f be the continuous function and take V and W to be $f^{-1}\big([0, \frac{1}{2})\big)$ and $f^{-1}\big((\frac{1}{2}, 1]\big)$ respectively. The proof that $\mathbf{T_4}$ implies $\mathbf{T_{3\frac{1}{2}}}$ is considerably more difficult and in fact relies on the following deep result:

URYSOHN'S LEMMA. *If X is a $\mathbf{T_1}$ space, then X is $\mathbf{T_4}$ if and only if for each pair of disjoint closed subspaces E, $F \subset X$ there is a continuous function $f : X \to [0,1]$ such that $f = 0$ on E and $f = 1$ on F.*

The proof of the ($\Longleftarrow$) implication is similar to the proof that $\mathbf{T_{3\frac{1}{2}}}$ implies $\mathbf{T_3}$; the reader should fill in the details.

The remaining implication is an immediate consequence of this result. We shall not need the result in this generality (a simple proof for metric spaces is given below); Section 33 of Munkres (pages 207–212) give a detailed proof.■

There are two things one would like for the preceding list of separation properties. First of all, every metric space should be a $\mathbf{T_4}$ space. Second, to avoid redundancies one would like to know that if $\mathbf{i} > \mathbf{j}$ then $\mathbf{T_i}$ and $\mathbf{T_j}$ are not logically equivalent; *i.e.*, there is an example of a topological space that is a $\mathbf{T_j}$ space but not a $\mathbf{T_i}$ space. Some examples appear in Munkres, and there are many other examples of this sort in the book by Steen and Seebach. The proof that metric spaces are $\mathbf{T_4}$ turns out to be fairly straightforward.

PROPOSITION. *Every metric space is a $\mathbf{T_4}$ space.*

PROOF. We have already shown that metric spaces are $\mathbf{T_1}$ (and even $\mathbf{T_2}$). To prove the normality condition of Urysohn's Lemma involving continuous functions, consider the function

$$f(x) = \frac{\mathbf{d}(x,E)}{\mathbf{d}(x,E) + \mathbf{d}(x,F)} \ .$$

We know that the distance functions in the formula are continuous, so the formula will define a continuous real valued function if the denominator is nonzero. But since E and F are disjoint it follows that for each $x \in X$ either $x \notin E$ or $x \notin F$ is true (and maybe both are true). This means that at least one of the numbers $\mathbf{d}(x,E)$, $\mathbf{d}(x,F)$ is positive and hence their sum is always positive. Since the numerator is nonnegative and less than or equal to the denominator, it follows that $f(x) \in [0,1]$ for all $x \in X$. If $x \in E$ then $\mathbf{d}(x,E) = 0$ and therefore $f(x) = 0$, while if $x \in F$ then $\mathbf{d}(x,F) = 0$ and therefore $f(x) = 1$.■

Compactness and separation axioms

The following result has appeared on many examinations. When combined with Urysohn's Lemma it provides a powerful means for constructing continuous real valued functions on compact Hausdorff spaces with far-reaching consequences, particularly in functional analysis. Stronger forms of this result exist for spaces that are $\mathbf{T_2}$ and *paracompact* (a condition implied by compactness; see Section 41 on pages 252–260 of Munkres, and particularly see Theorem 41.1 on pages 253–254 for an analog of the theorem below). The result for paracompactness also has far-reaching consequences in topology and differential geometry.

THEOREM. *If a topological space is compact and* $\mathbf{T_2}$*, it is also* $\mathbf{T_4}$*.*

Proof. We shall give a quick proof that uses Wallace's Theorem. The more traditional proof is given by combining Lemma 26.4 on page 166 of Munkres (with the reasoning given on the previous page), which shows that compact and $\mathbf{T_2}$ implies $\mathbf{T_3}$, with Theorem 32.3 on page 202 of Munkres, which shows that compact and $\mathbf{T_3}$ implies $\mathbf{T_4}$.

Recall that a space X is Hausdorff if and only if the diagonal Δ_X is closed in $X \times X$. If A and B are disjoint subsets of a set S, it is immediate that $A \cap B = \emptyset$ is true if and only if $(A \times B) \cap \Delta_X = \emptyset$. Combining these observations, we see that if E and F are disjoint closed subspaces of a Hausdorff space X then $E \times F$ is contained in the open set $X \times X - \Delta_X$.

If X is compact, then so are E and F, and therefore by Wallace's Theorem it follows that there are open subsets $U, V \subset X$ such that

$$E \times F \subset U \times V \subset X \times X - \Delta_X \ .$$

This means that U and V are disjoint open subsets containing E and F respectively.■

References for examples

As noted before, if $i < j$ it is not always easy to find examples of topological spaces that are $\mathbf{T_i}$ but not $\mathbf{T_j}$, and the examples in Munkres are spread out over Sections 31 through 33. Therefore we shall give an index to those examples here.

When analyzing the logical relations among separation properties it is often useful to consider the following related questions:

(1) *If A is a subspace of a* $\mathbf{T_i}$ *space, is A also a* $\mathbf{T_i}$ *space?*

(1) *If X and Y are* $\mathbf{T_i}$ *spaces, is their product $X \times Y$ also a* $\mathbf{T_i}$ *space?*

In particular, if $i < j$ and either **(1)** or **(2)** is true for $\mathbf{T_i}$ spaces but not for $\mathbf{T_j}$ spaces, then it follows that $\mathbf{T_i}$ is strictly weaker than $\mathbf{T_j}$. Specifically, the following answers to **(1)** and **(2)** show that $\mathbf{T_i}$ is strictly weaker than $\mathbf{T_4}$ if $i < 4$:

FACTS. *The answers to both* **(1)** *and* **(2)** *are positive for* $\mathbf{T_i}$ *spaces if $i < 4$ and negative for* $\mathbf{T_4}$ *spaces,*

Given that the proofs for $i < 4$ are fairly direct, the failures of these results to hold for $\mathbf{T_4}$ are a bit surprising at first. However, they become less surprising in light of Urysohn's Lemma, which has many far-reaching implications (compare the first paragraph of Section 32 in Munkres).

It is relatively straightforward to show that the answers to both **(1)** and **(2)** are positive for $\mathbf{T_0}$ and $\mathbf{T_1}$, and the proofs are left to the reader as exercises. It is also straightforward to produce examples to show that $\mathbf{T_0}$ does not imply $\mathbf{T_1}$ and $\mathbf{T_1}$ does not imply $\mathbf{T_2}$. In the first case, one can use the Sierpiński space whose underlying set is $\{0, 1\}$ and whose open subsets are the empty set, the set itself and $\{0\}$. In the second case one can use the finite complement topology on an infinite set. We shall now list the references to Munkres for the remaining cases.

Subspaces and products of $\mathbf{T_i}$ *spaces are* $\mathbf{T_i}$ *if $i = 1, 2, 3, 3\frac{1}{2}$.* — The first two cases are treated in Theorem 31,2 on pages 196–197 of Munkres, and the other case is treated in Theorem 33.2 on pages 211-212 of Munkres.∎

Subspaces and products of $\mathbf{T_4}$ *spaces are not necessarily* $\mathbf{T_4}$. — The reference here is a combination of Theorem 32.1 and Example 2 on pages 202–204 of Munkres.∎

$\mathbf{T_2}$ *does not imply* $\mathbf{T_3}$. — The reference is Example 1 on pages 197–198 of Munkres.∎

$\mathbf{T_3}$ *does not imply* $\mathbf{T_{3\frac{1}{2}}}$. — The reference is Exercise 11 on pages 214 of Munkres.∎

$\mathbf{T_{3\frac{1}{2}}}$ *does not imply* $\mathbf{T_4}$. — One can extract this from Example 2 on pages 203–204 either by taking a subspace of a $\mathbf{T_4}$ space that is not $\mathbf{T_4}$ or by taking a product of $\mathbf{T_4}$ spaces that is not $\mathbf{T_4}$. Since $\mathbf{T_4}$ implies $\mathbf{T_{3\frac{1}{2}}}$ and this property is preserved under taking subspaces and products, it follows that the spaces in the given example are $\mathbf{T_{3\frac{1}{2}}}$ but not $\mathbf{T_4}$.∎

As we have already noted, the book by Steen and Seebach is a comprehensive summary of many further results and examples for the logical interrelationships of various special properties of topological spaces.

Non-Hausdorff topologies

As noted in Munkres and numerous other references, the addition of the Hausdorff Separation Property to the axioms for a topological space yields a class of objects that are more general than metric spaces but are relatively closed to one's geometric intuition in many ways. One additional motivation is that most of the spaces that are important to mathematicians satisfy the Hausdorff Separation Property. In some respects the examples of non-Hausdorff spaces that one sees in an introductory topology course may be viewed as instructive, showing that non-Hausdorff spaces often have very strange properties and pointing out that in some cases this property does not automatically hold, even if one makes simple constructions starting with Hausdorff spaces.

However, there are mathematical situations in which non-Hausdorff spaces arise, and in some branches of mathematics these examples turn out to be extremely important. One particularly noteworthy class of examples is given by the Zariski topologies from algebraic geometry. Here is the basic idea in the most fundamental cases: Let $\mathbf{k}$ be an algebraically closed field (every nonconstant polynomial factors completely into a product of linear polynomials, as in the complex numbers). A set $A \subset \mathbf{k}^n$ is said to be *Zariski closed* if A is the set of solutions for some finite system or polynomial equations in n variables (where the coefficients lie in $\mathbf{k}$). It is an easy exercise in algebra to show that the family of all such subsets satisfies the conditions for closed subsets of a topological space (see nearly any textbook on algebraic geometry or commutative algebra), and the resulting topological structure is called the Zariski topology. Other objects in algebraic geometry admit similar notions of Zariski topologies, but the latter quickly reach beyond the scope of this course. If $n = 1$ then the Zariski topology on $\mathbf{k}$ is the finite complement topology; since algebraically closed fields are always infinite (this follows immediately from the theory of finite fields), the Zariski topology on $\mathbf{k}$ is not Hausdorff although it is $\mathbf{T_1}$. Similarly, most other Zariski topologies are not Hausdorff (and not necessarily even $\mathbf{T_1}$). Two of the exercises for earlier sections (one on irreducible spaces, one on noetherian spaces) contain elementary results that arise naturally when one works with Zariski topologies; in fact, $\mathbf{k}^n$ with the Zariski topology is both irreducible and noetherian. The introduction of these topological structures in the nineteen forties was an elementary but far-reaching step in formulating the present day mathematical foundations for algebraic geometry. Dieudonné's book on the history of algebraic geometry provides some further information on these points.

During the past 25 to 30 years, non-Hausdorff topological spaces have also been used in certain areas of theoretical computer science. Although many of the basic ideas in such studies come from topology and branches of the "foundations of mathematics," the basic structures of the work and its goals differ substantially from those of the traditional core of mathematics, and the relevant spaces are much less geometrically intuitive than the Zariski topologies.

Two introductory references for this material are Section 3.4 of the Book, *Practical Foundations of Mathematics*, by P. Taylor, and a survey article by M. W. Mislove, *Topology, Domain Theory and Theoretical Computer Science* (Topology Atlas Preprint #181), which is available online at `http://at.yorku.ca/p/a/a/z/15.htm~`. The book *Domains and Lambda Calculi*, by R. Amadio and P.-L. Curien, presents this material specifically in connection with its applications to topics

VI.4 : Local compactness and compactifications

(Munkres, §§ 29, 37, 38)

As in the case of connectedness, it is often useful to have a variant of compactness that reflects a basic property of open subsets in Euclidean spaces: Specifically, if U is such a set and $x \in U$, then one can find an $\varepsilon > 0$ such that the closure of $N_\varepsilon(x)$ is compact; in fact if we choose $\delta > 0$ such that $N_\delta(x) \subset U$, then the compact closure property will hold for all ε such that $\varepsilon < \delta$.

Definitions. A topological space X is said to be *locally compact in the weak sense* if for each $x \in X$ there is a compact subset N such that $x \in \mathrm{Int}(N) \subset N$.

A topological space X is said to be *locally compact in the strong sense* if for each $x \in X$ and each open subset U containing x there is a compact subset N such that $x \in \mathrm{Int}(N) \subset N \subset U$.

Clearly the second condition implies the first, but in many important cases these concepts are equivalent:

PROPOSITION. *If X is a Hausdorff space, then X is locally compact in the weak sense if and only if X is locally compact in the strong sense. Furthermore, in this case if $x \in X$ and U is an open set containing x, then there is an open set W such that $x \in W \subset \overline{W} \subset U$ and $\overline{W}$ is compact.*

Proof. It is only necessary to show that the weak sense implies the strong sense and that the additional condition holds in this case. Suppose that X is locally compact in the weak sense and that $x \in U$ where U is open in X. Let N be the compact set described above, and let $V = \mathrm{Int}(N) \cap U$. Then $x \in V$ where V is open in N. Since the latter is regular (it is compact Hausdorff), there is an open subset $W \subset V$ such that

$$x \in W \subset \mathrm{Closure}(W, N) \subset V \ .$$

The set W is in fact open in X (because $W = N \cap W'$ where W' is open in X and $W = W \cap V = V \cap W'$ since $W \subset V \subset N$), Furthermore, since N is closed in X it follows that $\mathrm{Closure}(W, N) = \overline{W} \cap N$ must be equal to $\overline{W}$.

Not all subspaces of a locally compact Hausdorff space are locally compact. For example, the set of all rational numbers in the real line is not locally compact. (**Proof:** Suppose that $a \in \mathbf{Q}$ and that $B \subset \mathbf{Q}$ is an open subset in the subspace topology such that $\overline{B} \cap \mathbf{Q}$ is compact. Without loss of generality we may assume that B is an open interval centered at a. The compactness assumption on the closure implies that $\overline{B} \cap \mathbf{Q}$ is in fact a compact, hence closed and bounded, subset of the real line. This is impossible since there are many irrational numbers that are limit points of B.). However, a large number of interesting subspaces are locally compact.

PROPOSITION. *If X is a locally compact Hausdorff space in the strong sense and Y is* **either** *an open or a closed subset of X, then Y is locally compact (and Hausdorff).*

Proof. The proof for open subsets follows because if Y is open in X and U is open in Y, then U is open in X; one can then use the strong form of local compactness to prove the existence of an open subset of U with the required properties.

Suppose now that Y is closed, let $y \in Y$, and let U be an open subset of Y containing y. Write $U = Y \cap U_1$ where U_1 is open in X. Then there is an open set W_1 in X such that

$$x \ \in \ W_1 \ \subset \ \overline{W_1} \ \subset \ U_1$$

and $\overline{W_1}$ is compact. Let $W = W_1 \cap Y$; then

$$x \in W \subset \overline{W_1} \cap Y \subset U$$

where $\overline{W_1} \cap Y$ is compact because it is closed in X and contained in the compact subspace $\overline{W_1}$. Since

$$\text{Closure}(W, Y) = \overline{W} \cap Y \subset \overline{W_1} \cap Y$$

it follows that Closure(W, Y) is compact and that

$$x \in W \subset \text{Closure}(W, Y) \subset \overline{W_1} \cap Y \subset U$$

and therefore Y is locally compact in the strong sense.

COROLLARY. *If X is locally compact Hausdorff and $B = U \cap F$, where $U \subset X$ is open and $F \subset X$ is closed, then B is locally compact.*

Proof. By the proposition we know that F is locally compact Hausdorff. Since $B = U \cap F$ is open in F, the proposition then implies the same conclusion for B.■

COROLLARY. *If U is open in a compact Hausdorff space, then U is locally compact.*

Proof. A compact Hausdorff space is clearly locally compact in the weak sense and hence locally compact in the strong sense. Therefore U must also be locally compact in the strong sense.

COROLLARY. *If X is a locally compact Hausdorff space, then X is* $\mathbf{T_3}$.

The defining conditions for locally compact Hausdorff in the strong sense imply that such a space is regular. In fact, one can go further using Urysohn's Lemma to prove that a locally compact Hausdorff space is in fact completely regular (hence it is $\mathbf{T_{3\frac{1}{2}}}$).

Note. A locally compact Hausdorff space is not necessarily $\mathbf{T_4}$. For example, Example 2 on pages 203–204 of Munkres actually describes an OPEN subset of a compact Hausdorff space that is not $\mathbf{T_4}$, and by the proposition above this subset is locally compact Hausdorff.

Compactifications of noncompact spaces

Frequently in mathematics it is helpful to add points at infinity to a mathematical system to deal with exceptional cases. For example, when dealing with limits in single variable calculus this can be done using an extended real number system that consists of the real line together with two additional points called $\pm\infty$. In some other cases, it is preferable to add only a single point at infinity; for example, this is necessary if one wants to have something equal to

$$\lim_{x \to 0} \frac{1}{x}$$

and in the theory of functions of a complex variable it is also natural to have only one point at infinity.

In other situations it is desirable to add many different points at infinity. Projective geometry is perhaps the most basic example. In this subject one wants to add a point at infinity to each line in such a way that two lines are parallel if and only if their associated extended lines contain the same point at infinity. This turns out to be useful for many reasons; in particular, it allows one to state certain results in a uniform manner without detailed and often lengthy lists of special cases (however, you do not actually get something for nothing — one must first invest effort into the construction of points at infinity in order to obtain simplified arguments and conclusions).

In all these cases and many others, one basic property of the enriched spaces with added points at infinity is that these enriched spaces are compact and contain the original spaces as dense subspaces. In order to simplify the discussion but include the most interesting examples, we shall only consider (original and enriched) spaces that are Hausdorff.

Definition. If X is a topological space, then a *compactification* of X is a pair (Y, f) where Y is compact and $f : X \to Y$ is a continuous map that is a homeomorphism onto a dense subspace. Two compactifications (Y, f) and (Z, g) are *equivalent* if there is a homeomorphism $h : Y \to Z$ such that $h \circ f = g$.

The compactification (Y, f) is said to *dominate* the compactification (Z, g) if there is a continuous map $h : Y \to Z$ such that $h \circ f = g$, and in this case we write $(Y, f) \geq (Z, g)$.

There is also a corresponding notion of *abstract closure* in which there is no compactness assumption on Y, and one can define equivalence and domination in a parallel manner. Given an abstract closure (Y, f) of X, its *residual set* is the subset $Y - f(X)$.

Here are some basic properties of Hausdorff compactifications and abstract closures.

PROPOSITION. *(i) If (Y, f) and (Z, g) are Hausdorff compactifications or abstract closures of the same Hausdorff space X, then there is at most one $h : Y \to Z$ such that $h \circ f = g$.*

(ii) If (Y, f) and (Z, g) are Hausdorff compactifications such that $(Y, f) \geq (Z, g)$ and h is the continuous map defining the domination, then h is onto.

(iii) There is a set of equivalence classes of Hausdorff compactifications or abstract closures of a Hausdorff spaces such that every Hausdorff compactification of X is equivalent to a compactification in that set.

(iv) The relation of domination makes the equivalence classes of Hausdorff compactifications or abstract closures into a partially ordered set.

PROOF. *Proof of (i).* Let $h : Y \to Z$ be a homeomorphism such that $h \circ f = g$. If h' is another such map then the restrictions of h and h' to $f(X)$ are equal. Since $f(X)$ is dense and the set of points where two functions into a Hausdorff space are equal is a closed subset of the domain, it follows that this subset is all of Y and thus $h = h'$ everywhere.

Proof of (ii). By construction the image of h contains the dense subspace $g(X)$, and since $h(Y)$ is compact and Z is Hausdorff it also follows that $h(Y)$ is closed. Therefore we must have $h(Y) = Z$.

Note that the analogous result for abstract closures is false; by definition the identity map on a Hausdorff space X is an abstract closure in the sense of the definition, and in general there are many abstract closures (Y, f) such that $f(X) \neq Y$.

Proof of (iii). We need to introduce some set-theoretic notation. Given a set S and a nonempty family of subsets $\mathcal{M} \subset \mathbf{P}(S)$, a family $\mathcal{A} \subset \mathbf{P}(S)$ is called a *filter* provided

[*a*] if $B \in \mathcal{A}$, $B \subset C$ and $C \in \mathcal{M}$ then $C \in \mathcal{A}$,

[*b*] if $B \in \mathcal{A}$ and $C \subset \mathcal{A}$ then $B \cap C \in \mathcal{A}$.

Formally, this concept is dual to the concept of an ideal in a Boolean algebra where union and intersection are interpreted as addition and multiplication, but for our purposes the important point is that in a topological space X the set $\mathcal{N}_x$ of all open subsets containing a given point $x \in X$ is a filter.

Given a family of subsets $\mathcal{B} \subset \mathcal{M}$ that is closed under finite intersections, the smallest filter $\mathcal{B}^*$ containing $\mathcal{B}$ (or the filter generated by $\mathcal{B}$) consists of $\mathcal{B}$ together with all subsets containing some element of $\mathcal{B}$.

If X is a Hausdorff topological space then $\mathcal{F}(X)$ will denote the set of all filters of open subsets in X. For Hausdorff spaces it is immediate that $p \neq q$ implies $\mathcal{N}_p \neq \mathcal{N}_q$. Given a 1–1 continuous map g from a Hausdorff space X to another Hausdorff space Y, there is an associated map f^* from Y to $\mathcal{F}(X)$ that sends y to the filter generated by $f^{*-1}(\mathcal{N}_y)$.

We claim that f^* is 1–1 provided f is 1–1, Y is Hausdorff and $f(X)$ is dense in Y. Suppose that $u, v \in Y$. Then there are disjoint open subset $U, V \in Y$ such that $u \in U$ and $v \in V$ It suffices to show that $f^{-1}(U) \in f^*(u) - f^*(v)$ and $f^{-1}(V) \in f^*(v) - f^*(u)$. In fact, if we can prove the first, then we can obtain a proof of the second by reversing the roles of u and v and U and V throughout the argument.

By construction we have $f^{-1}(U) \in f^*(u)$ so verification of the claim reduces to showing that $f^{-1}(U) \notin f^*(v)$. Assume that we do have $f^{-1}(U) \in f^*(v)$. Then there is some open set W in Y containing v such that $f^{-1}(W) \subset f^{-1}(U)$, and it follows that we also have $f^{-1}(W \cap V) \subset f^{-1}(U)$. Since $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$ this means that $f^{-1}(W \cap V)$ must be empty. To see that this is impossible, note that $f(X) \cap W \cap V \neq \emptyset$ because $v \in W \cap V$ and $f(X)$ is dense in Y.

It follows that Y can be identified with a subset of $\mathcal{F}(X)$. By construction the latter is a subset of $\mathbf{P}(X)$ and therefore the number of points in Y is at most the cardinality of $\mathbf{P}(\mathbf{P}(X))$. Since each topology on Y is a family of subsets of Y it follows that there is a specific bound on the cardinality of equivalence classes of Hausdorff spaces that contain a homeomorphic copy of X as a dense subspace.

Proof of (iv). Reflexivity is trivial (take the identity map on the compact space) and transitivity is also trivial (take the composite of the mappings on the compact spaces). Suppose now that $(Y, f) \geq (Z, g)$ and vice versa. Then there are continuous maps $h : Y \to Z$ and $k : Z \to Y$ such that $g = h \circ f$ and $f = k \circ g$. Therefore we have $f = k \circ h \circ f$ and $g = h \circ k \circ g$, so that $k \circ h$ and id_Y agree on the dense subset $f(X)$ and $h \circ k$ and id_Z agree on the dense subset $g(X)$. These imply that h and k are inverses to each other and hence that the two compactifications are equivalent.■

The Alexandroff one point compactification

In general there are many compactifications of a Hausdorff space. For example, there are many closed bounded subsets of the plane that contain open dense subsets homeomorphic to $\mathbf{R}^2 \cong (0,1)^2$ (these include solid rectangles with an arbitrary finite number of open holes removed; see any of the files `swisscheese.*` for more about this). Furthermore, a Hausdorff compactification does not necessarily inherit certain "good" properties of the original space; in particular, Example 3 on page 238 of Munkres, shows that a compactification of the real line is not necessarily locally connected or path connected.

In view of the examples in the previous paragraph and many others, it seems advisable to begin with simple questions about the structure of the partially ordered set of equivalence classes of Hausdorff compactifications. Two obvious questions are whether this set has maximal or minimal elements. It turns out that every $\mathbf{T_{3\frac{1}{2}}}$ space has a maximal compactification (the Stone-Čech compactification) that is unique up to equivalence. This object is constructed directly in Section 38 of Munkres.

Note. Since a compact $\mathbf{T_2}$ space is $\mathbf{T_4}$ and every subspace of a $\mathbf{T_{3\frac{1}{2}}}$ space is again $\mathbf{T_{3\frac{1}{2}}}$, it follows that a topological space has a Hausdorff compactification if and only if it is $\mathbf{T_{3\frac{1}{2}}}$.

Our main purpose here is to consider the minimal type of compactification where the added set $Y - f(X)$ consists of a single point. If $X = \mathbf{R}^n$ there is a standard and important visualization of this compactification; Y turns out to be homeomorphic to the n-dimensional unit sphere in $\mathbf{R}^{n+1}$.

If a space X has a Hausdorff compactification (Y, f) such that $Y - f(X)$ is a single point (or more generally a closed subset!) then by a corollary stated above the space X must be locally compact. This will explain our assumption in the basic construction.

Definition. Let X be a locally compact Hausdorff topological space that is not compact, and let ∞ be some point not in X (for example, in the standard axiomatic model for set theory one has $X \notin X$ so we can take $\infty = \{X\}$). The *one point* or *Alexandroff* compactification of X is the set $X^\bullet = X \cup \{\infty\}$ with open sets given as follows:

(1) *"Bounded open sets"* that are open subsets of X itself.
(2) *"Open neighborhoods of* ∞*"* that are unions $\{\infty\} \cup X - K$ where K is a compact subset of X.

It is necessary to verify that this family of sets defines a topology on X. The empty set is open in $X^\bullet$ because it is open in X, and $X^\bullet$ is open because it is equal to $\{\infty\} \cup X - \emptyset$ and $\emptyset$ is compact. Suppose that we have a family of open sets in $X^\bullet$, and split it into the bounded open sets U_α and the open neighborhoods of infinity $\{\infty\} \cup X - K_\beta$ where K_β is compact. Some elementary set-theoretic manipulation shows that the union of this family is either the bounded open set $\cap_\alpha U_\alpha$ if there are no open neighborhoods of infinity in the family or else it is

$$\{\infty\} \bigcup X - (K - U)$$

where $U = \cup_\alpha U_\alpha$ and $K = \cap_\beta K_\beta$ (note that an arbitrary intersection of compact subsets in a Hausdorff space is compact). The details of the set-theoretic algebra are described on the top of page 184 of Munkres. The latter also gives the arguments to verify that the family of open subsets defined above is closed under (finite) intersection.

CLAIM. *The inclusion of X in $X^\bullet$ is a compactification.*

Indication of proof. We need to show that the inclusion of X in $X^\bullet$ is 1–1 continuous and open (this will show it is a homeomorphism onto its image), that the image of X is dense in $X^\bullet$ and that $X^\bullet$ is compact Hausdorff. A proof that $X^\bullet$ is compact Hausdorff appears near the bottom of page 184 of Munkres.

To see that X is dense in $X^\bullet$, it suffices to verify that ∞ is a limit point of X. Let

$$\{\infty\} \bigcup (X - K)$$

(with K compact) be an open set containing the point at infinity. Since X is noncompact we must have $X - K \neq \emptyset$, and this proves the limit point assertion.

By construction the inclusion map from X to $X^\bullet$ is 1–1 and open; we need to show it is also continuous. This follows because the inverse image of a finite open set is just the set itself, and the latter is open in X by construction, while the inverse image of a open neighborhood of infinity has the form $X - K$ where K is compact; since compact subsets are closed it follows that the inverse image $X - K$ is open in X.■

The uniqueness of this one point compactification is given by the following result:

PROPOSITION. *If X is locally compact Hausdorff and (Y, f) is a Hausdorff compactification such that $Y - f(X)$ is a single point, then there is a unique homeomorphism $h : X^\bullet \to Y$ such that $h|X = f$.*

The proof of this is given as Step 1 on the bottom of page 183 in Munkres.∎

One point compactifications of Euclidean spaces

Since the spaces $\mathbf{R}^n$ are some of the most fundamental examples of locally compact spaces that are not compact, it is natural to ask if some additional insight into the nature of the one point compactification can be obtained in these cases. The following result shows that the one point compactification of $\mathbf{R}^n$ is homeomorphic to the n-dimensional sphere

$$S^n = \{ \, x \in \mathbf{R}^{n+1} \mid |x|^2 = 1 \, \} \, .$$

PROPOSITION. *Let $\mathbf{e}_{n+1} \in \mathbf{R}^{n+1}$ be the unit vector whose last coordinate is 1 (and whose other coordinates are zero). Then there is a canonical homeomorphism from the subspace $S^n - \{ \mathbf{e}_{n+1} \}$ to $\mathbf{R}^n$.*

Sketch of proof. The homeomorphism is defined by *stereographic projection*, whose physical realization is the polar projection map of the earth centered at the south pole. Mathematically this is given as follows: View $\mathbf{R}^n$ as the linear subspace spanned by the first n unit vectors, and given a point $v \in S^n$ such that $v \neq \mathbf{e}_{n+1}$ let w be the unique point of $\mathbf{R}^n$ such that $w - \mathbf{e}_{n+1}$ lies on the straight line joining $\mathbf{e}_{n+1}$ to v. The explicit formula for this map is

$$f(v) = 2\,\mathbf{e}_{n+1} \;+\; \frac{2}{1 - \langle v, \mathbf{e}_{n+1} \rangle} \cdot \big(v - \mathbf{e}_{n+1}\big)$$

and illustrations of this appear in the files `sterepic2.*` in the course directory.

The stereographic projection map f is continuous by the formula given above, and elementary considerations from Euclidean geometry show that this map defines a 1–1 correspondence between $S^n - \{ \mathbf{e}_{n+1} \}$ and $\mathbf{R}^n$. In order to give a rigorous proof that f is a homeomorphism, it suffices to verify that the map

$$g : \mathbf{R}^n \to S^n - \{ \mathbf{e}_{n+1} \}$$

defined by the formula

$$g(w) = \mathbf{e}_{n+1} = \frac{4}{|w|^2 + 4} \cdot \big(w - 2\,\mathbf{e}_{n+1}\big)$$

is an inverse to f; *i.e.*, we have $g(f(v)) = v$ and $f(g(w)) = w$ for all $v \in S^n - \{ \mathbf{e}_{n+1} \}$ and $w \in \mathbf{R}^n$. In principle the verification of these formulas is entirely elementary, but the details are tedious and therefore omitted.∎

The following important geometrical property of stereographic projection was essentially first established by Hipparchus of Rhodes (*c.* 190 B.C.E. – *c.* 120 B.C.E.):

CONFORMAL MAPPING PROPERTY. *Let $\alpha, \beta : [0,1] \to \mathbf{R}^n$ be differentiable curves with $\alpha(0) = \beta(0)$ and $\alpha'(0), \beta'(0) \neq 0$. Then the image curves $g \circ \alpha$ and $g \circ \beta$ in S^n satisfy the conditions $g \circ \alpha(0) = g \circ \beta(0)$ and $[g \circ \alpha]'(0), [g \circ \beta]'(0) \neq 0$, and*

$$\mathbf{angle}\big(\alpha'(0), \beta'(0)\big) = \mathbf{angle}\big([g \circ \alpha]'(0), [g \circ \beta]'(0)\big) \, .$$

This result will be established in the appendix on stereographic projection and inverse geometry.

VI.5 : Metrization theorems

(Munkres, §§ 39, 40, 41, 42)

As noted before, it is natural to ask for necessary and sufficient conditions that a topology on a space comes from a metric. Most point set topology texts, including Munkres, cover this material in considerable detail.

Our approach here is somewhat different; namely, we want to show that compact Hausdorff spaces built out of compact subsets of finite-dimensional Euclidean spaces are also homeomorphic to subsets of such spaces. Examples of such objects arise in many geometric and topological contexts. Our results deal with subsets of finite-dimensional Euclidean spaces and the general metrization results involve finding homeomorphic copies of a space in various infinite-dimensional spaces, so the results given here are not actually contained in the more general ones that are stated and proved in Munkres.

Since a metric is by definition a real valued continuous function on $X \times X$, it is not surprising that the proofs of metrization theorems rely heavily on constructing continuous real valued functions on a space. Therefore it is necessary to begin with results about continuous functions on compact metric spaces.

Constructions for continuous functions

The basis for all these constructions is *Urysohn's Lemma*, which is true for arbitrary $\mathbf{T_4}$ spaces and was verified earlier in the notes for metric spaces: *Given two nonempty disjoint closed subsets E and F in X, there is a continuous function $f : X \to [0,1]$ such that $f = 0$ on E and $f = 1$ on F.*

Notation. A Hausdorff space will be called a **UL**-space if Urysohn's Lemma is true. We have already noted that this is equivalent to the $\mathbf{T_4}$ condition and that every metric space is a **UL**-space.

The following basic result is in fact logically equivalent to Urysohn's Lemma:

TIETZE EXTENSION THEOREM. *Let X be a* **UL***-space, let A be a closed subset of X and let $f : A \to [0,1]$ be a continuous function. Then f extends to a continuous function from all of X to $[0,1]$.*

It is easy to prove a converse result that X is a **UL**-space if for every closed subset $A \subset X$ and every continuous function $A \to [0,1]$ there is an extension to X, for if E and F are closed subsets of X, then the function that is 0 on E and 1 on F is continuous and defined on a closed subset; the extension to X shows that the **UL**-space condition is fulfilled.

Proof. Since the closed intervals $[0,1]$ and $[-1,1]$ are homeomorphic, we may as well replace the former by the latter in the proof. The main idea of the proof is to construct a sequence of functions $\{ \varphi_n \}$ such that

$$\lim_{n\to\infty} \varphi_n | A = f$$

in $\mathbf{BC}(A)$, and the basis for the recursive step in the construction is the following:

LEMMA. *Let $r > 0$ and let $h : A \to [-r, r]$ be continuous. Then there is a continuous function $g : X \to \left[-\frac{1}{3}r, \frac{1}{3}r\right]$ such that $\| \, g|A - h \, \| \leq \frac{2}{3}r$.*

Proof of Lemma. Let $B = h^{-1}\big([-r, -\frac{1}{3}r]\big)$ and $C = h^{-1}\big([\frac{1}{3}r, r]\big)$, so that B and C are disjoint closed subsets of A. Take $g : X \to \big[-\frac{1}{3}r, \frac{1}{3}r\big]$ to be a continuous function such that $g = -\frac{1}{3}r$ on B and $g = \frac{1}{3}r$ on C. The verification that $\| g|A - h \| \leq \frac{2}{3}r$ separates into three cases depending upon whether a point $a \in A$ lies in B, C or

$$h^{-1}\big([-\tfrac{1}{3}r, \tfrac{1}{3}r]\big)$$

(at least one of these must hold). In each case one can show directly that $|g(a) - h(a)| \leq \frac{2}{3}r$.∎

Proof of the Tietze Extension Theorem continued. Start off with $f : A \to [-1, 1]$, and apply the lemma to get a function $g_1 : X \to [-\frac{1}{3}, \frac{1}{3}]$ with the properties stated in the lemma. Consider now the function $f_1 = f - (g_1|A)$, which is a continuous function that takes values in $[-\frac{2}{3}, \frac{2}{3}]$. Let g_2 be the continuous function on X associated to f_1 as in the lemma, define f_2 to be $f_1 - (g_2|A)$, and note that

$$\| f_2 \| \leq \left(\frac{2}{3}\right)^2.$$

We can now continue recursively to define sequences of continuous real valued functions g_n on X and $f_n = f_{n-1} - (g_n|A)$ such that

$$\| g_n \| \ \leq \ \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{n-1}, \quad \| f_n \| \ \leq \ \left(\frac{2}{3}\right)^{n-1}.$$

Note that we have

$$f_n = f - \sum_{i=1}^{n} g_n|A$$

for all positive integers n.

We want to define a continuous function $G(x)$ by an infinite series

$$\sum_n g_n$$

and this will be possible if

$$\sum_n \| g_n \|$$

converges. But the latter sum is dominated by the convergent series

$$\frac{1}{3} \cdot \sum_n \left(\frac{2}{3}\right)^{n-1}$$

so there is no problem with constructing the continuous function G. To see that $G|A = f$, note that $G|A = \sum_n g_n|A$ and for each n we have that

$$\| f - \sum_{i=1}^{n} (g_i|A) \| \ = \ \| f_n \| \ \leq \ \left(\frac{2}{3}\right)^n$$

so that we also have $f = \sum_n g_n|A$. Finally, G maps X into $[-1, 1]$ because

$$\| G \| \ \leq \ \frac{1}{3} \cdot \sum_n \left(\frac{2}{3}\right)^{n-1} \ = \ 1 .$$ ∎

COROLLARY. *Let X be a* **UL***-space, let A be a closed subset of X and let $f : A \to (-1,1)$ be a continuous function. Then f extends to a continuous function from all of X to $(-1,1)$.*

Proof. By the theorem we have a continuous function $G : X \to [-1,1]$ such that $G = f$ on A. We need to modify this function to something that still extends f but only takes values in $(-1,1)$.

Let $D \subset X$ be $G^{-1}(\{-1,1\})$. By construction D and A are disjoint closed subsets, so there is a continuous function $k : X \to [0,1]$ that is 0 on D and 1 on A. If we set F equal to the product of G and k, then it follows that F takes values in $(-1,1)$ and $F|A = f$.■

COROLLARY. *Let X be a* **UL***-space, let A be a closed subset of X and let $f : A \to \mathbf{R}^n$ be a continuous function. Then f extends to a continuous function from all of X to $\mathbf{R}^n$.*

Proof. If $n = 1$ this follows from the previous corollary because $(-1,1)$ is homeomorphic to $\mathbf{R}$ (specifically, take the map $h(x) = x/(1-|x|)$). If $n \geq 2$ let $f_1, \cdots, f_n$ be the coordinate functions of f, and let F_i be a continuous extension of f_i for each i. If F is defined by the formula

$$F(x) = \big(F_1(x), \cdots, F_n(x)\big)$$

then $F|A = f$.■

Piecewise metrizable spaces

The following result provides a useful criterion for recognizing that certain compact Hausdorff spaces that are homeomorphic to subsets of Euclidean spaces.

PROPOSITION. *Let X be a compact Hausdorff space such that X is a union of closed subsets $A \cup B$, where A and B are homeomorphic to subsets of some finite-dimensional Euclidean space(s). Then X is also homeomorphic to a subset of some finite-dimensional Euclidean space.*

Proof. We may as well assume that both A and B are homeomorphic to subsets of the same Euclidean space $\mathbf{R}^n$ (take the larger of the dimensions of the spaces containing A and B respectively). Let $f : A \to \mathbf{R}^n$ and $g : B \to \mathbf{R}^n$ be 1–1 continuous mappings (hence homeomorphisms onto their images).

Let $F_0 : B \to \mathbf{R}^n$ and $G : A \to \mathbf{R}^n$ be continuous functions that extend $f|A \cap B$ and $g|A \cap B$ respectively. We can then define $F, G : X \to \mathbf{R}^n$ by piecing together f and F_0 on A and B in the first case and by piecing together G_0 and g on A and B in the second.

We shall also need two more continuous functions to construct a continuous embedding (a 1–1 continuous map that is a homeomorphism onto its image) on $X = A \cup B$. Let $\alpha : X \to \mathbf{R}$ be defined by $\mathbf{d}_B(x, A \cap B)$ on B and by 0 on A; this function is well defined and continuous because the two definitions agree on $A \cap B$. Note also that $\alpha(x) = 0$ if and only if $x \in A$. Similarly, let $\beta : X \to \mathbf{R}$ be defined by $\mathbf{d}_A(x, A \cap B)$ on A and by 0 on B; as before, this function is well defined, and furthermore $\beta(x) = 0$ if and only if $x \in B$.

Define a continuous function

$$h : X \to \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \cong \mathbf{R}^{2n+2}$$

by the formula

$$h(x) = \big(F(x), G(x), \alpha(x), \beta(x)\big) \ .$$

By construction h is continuous, and since X is compact the map h will be an embedding if and only if h is 1–1. Suppose that $h(y) = h(z)$. If $y \in A$ then we know that $\alpha(y) = 0$ and therefore we must also have $\alpha(z) = 0$ so that y and z both belong to A. Likewise, if $h(y) = h(z)$ and $y \in B$ then we must also have $z \in B$. We then also have $F(y) = F(z)$ and $G(y) = G(z)$. If $y \in A$, then the fact that z also lies in A combines with the first equation to show that $y = z$, while if $y \in B$, then the fact that z also lies in B combines with the first equation to show that $y = z$. In either case we have that $h(y) = h(z)$ implies $y = z$.∎

COROLLARY. *If X is a compact Hausdorff space that is a finite union of the closed metrizable subspaces A_i that are homeomorphic to subsets of some finite-dimensional Euclidean space, then X is also homeomorphic to subsets of some finite-dimensional Euclidean space and hence metrizable.*∎

In particular, this gives an alternate proof of Theorem 36.2 on pages 226–227 of Munkres (the details are left to the reader, but here is a hint — show that the space in the theorem is a finite union of subspaces homeomorphic to closed disks in $\mathbf{R}^n$).∎

The next result is also useful for showing that certain compact Hausdorff spaces are homeomorphic to subsets of Euclidean spaces. Some preliminaries are needed.

Definition. If X and Y are topological spaces, and $f : X \to Y$ is continuous, then the *mapping cylinder* $\mathcal{M}_f$ of f is the quotient of $Y \amalg (X \times [0,1])$ modulo the equivalence relation generated by the condition

$$(x, 1) \in X \times \{1\} \quad \sim \quad f(x) \in Y \ .$$

The equivalence classes of this relation are the one point sets $\{ (x,t) \}$ for $t < 1$ and the hybrid sets

$$\{y\} \amalg f^{-1}\big(\{y\}\big) \times \{1\} \ .$$

This space is a quotient of a compact space (the disjoint union of two compact spaces) and therefore is compact.

PROPOSITION. *If X and Y are homeomorphic to subspaces of $\mathbf{R}^n$ for some n, then the mapping cylinder $\mathcal{M}_f$ is also homeomorphic to a subspace of some Euclidean space.*

Outline of proof. The details are left to the reader as an exercise, but the underlying idea is as follows. Start out with embeddings α and β of X and Y in $\mathbf{R}^n$ and construct a map

$$H : \mathcal{M}_f \to \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \cong \mathbf{R}^{2n+1}$$

that is equal to $(0, 0, \beta(y)\,)$ on Y and given by

$$\big(\,(1-t)\alpha(x), 1-t, t\cdot\beta(f(x))\,\big)$$

on $X \times [0,1]$. This yields a well defined map on $\mathcal{M}_f$ because it is consistent with the equivalence relation, and the proof that h is a homeomorphism onto its image reduces to showing that h is 1–1; the latter is an elementary exercise.∎

Example. In algebraic topology one often encounters the following construction called *adjoining or attaching a k-cell*: Given a space A and a continuous map $f : S^{k-1} \to A$, we define

$$B = A \cup_f e^k$$

to be the disjoint union of A and the disk D^k modulo the equivalence relation generated by $x \in S^{k-1}$ with $f(x) \in A$ for all x. Let $E \subset B$ be the set of all points that come from $A \cup \{\, x \in D^k \mid |x| \geq \frac{1}{2} \,\}$

and let F be the image of $\frac{1}{2}D^k$ in B. Then E is homeomorphic to $\mathcal{M}_f$, and then one can apply both results above to show that B is homeomorphic to a subset of some Euclidean space if A is homeomorphic to a compact subset of some Euclidean space.

A *finite cell complex* (also called a finite CW complex) is a compact Hausdorff space that is obtained from a one point space by a finite number of attachings of k_i-cells for varying values of i. We allow the case where $i = 0$ with the conventions that $S^{-1} = \emptyset$ and adjoining a 0-cell is simply the disjoint union of the original space with a one point space. The preceding results imply that each finite cell complex is homeomorphic to a subset of some Euclidean space.

An alternate approach

The metrizability proofs given above are direct and complete, and they show that the examples are in fact homeomorphic to subsets of ordinary finite dimensional Euclidean spaces. Another approach to proving the metrizability of quotients of compact metrizable spaces is by means of the *Hausdorff metric* on the family of closed subsets. The precise definitions and formal properties of the Hausdorff metric are given in Exercises 7 and 8 on pages 280–281 of Munkres. As noted in part (a) of Exercise 7, this metric makes the set of all closed subsets of a metric space X into another metric space that Munkres denotes by $\mathcal{H}$.

PROPOSITION. *Let X be a compact metric space, and let Y be a $\mathbf{T_1}$ quotient space of X. Then Y is metrizable.*

The two classes of examples above are homeomorphic to quotient spaces of compact metric spaces (presented as disjoint unions of other compact metric spaces), and in each case every equivalence class is a closed subset of the disjoint union. It follows that the quotient spaces are $\mathbf{T_1}$ in this case and therefore the proposition implies metrizability of the space constructed from the pieces (although it does not yield the embeddability in some Euclidean space if each piece is so embeddable). The need for the $\mathbf{T_1}$ condition is illustrated by a variant of an earlier example: Take $X = [-1, 1]$ and consider the equivalence relation $x \sim y$ if and only if x and y are positive real multiples of each other; as in Section 1 of Unit V, the quotient space is a non-Hausdorff space consisting of three points (in fact, this quotient space is not $\mathbf{T_1}$ because the equivalence classes of -1 and $+1$ are not closed subsets).

Proof of Proposition. If Y is $\mathbf{T_1}$ then every equivalence class in X is a closed subset, and therefore we have a map $F : X \to \mathcal{H}$ sending x to $f^{-1}(\{f(x)\})$. If we can show that F is continuous, then the metrizability of Y may be established as follows: Let $\pi : X \to Y$ be the quotient projection. Since $\pi(u) = \pi(v)$ implies $F(u) = F(v)$ there is a unique continuous map $g : Y \to \mathcal{H}$ such that $F = g \circ \pi$. By construction g is a 1–1 continuous map from the compact space Y (recall it is the image of a compact space) to the metric space $\mathcal{H}$, and therefore g is a homeomorphism onto its image.

To verify that F is continuous, let $\varepsilon > 0$, let $\mathbf{D}$ denote the Hausdorff metric on $\mathcal{H}$. What does it mean to say that $\mathbf{D}(F(u), F(v)) < \varepsilon$? If one defines $U(A, \varepsilon) \subset X$ as in Munkres to be the set of all points whosed distance from a subset A is less than ε, then the contidition on the Hausdorff metric is that $F(u) \subset U(F(v), \varepsilon)$ and $F(v) \subset U(F(u), \varepsilon)$. Suppose now that $\mathbf{d}(u, v) < \varepsilon$ where as usual $\mathbf{d}$ denotes the original metric on X. Then the distance from u to $F(v)$ is less than ε and likewise the distance from v to $F(u)$ is less than ε. Therefore we have $F(u) \subset U(F(v), \varepsilon)$ and $F(v) \subset U(F(u), \varepsilon)$, and as noted above this implies $\mathbf{D}(F(u), F(v)) < \varepsilon$ so that F is uniformly continuous (in fact, Lipschitz). We can now use the argument of the first paragraph of the proof to show that Y is metrizable.■

We begin with an early and powerful result. The proof is given in Theorem 34.1 on pages 215–218 of Munkres, with important preliminary material appearing in Theorem 20.5 on pages 125–126 and Theorem 32.1 on pages 200–201.

URYSOHN METRIZATION THEOREM. *Let X be a second countable space. Then X is homeomorphic to a metric space if and only if X is* $\mathbf{T_3}$.■

In fact, the argument shows that X is homeomorphic to a subspace of a compact metric space if and only if X is $\mathbf{T_3}$ and second countable, for one constructs an embedding into a countable product of copies of $[0,1]$ and the latter is compact by Tychonoff's Theorem (alternatively, one can use Exercise 1 on page 280 of Munkres to prove compactness of the product).

The necessary and sufficient conditions for the metrizability of arbitrary topological spaces require some additional concepts.

Definition. A family of subsets $\mathcal{A} = \{ A_\alpha \}$ in a topological space X is said to be *locally finite* if for each $x \in X$ there is an open neighborhood U such that $U \cap A_\alpha \neq \emptyset$ for only finitely many A_α.

Examples. Aside from finite families, perhaps the most basic examples of locally finite families are given by the following families of subsets of $\mathbf{R}^n$.

(1) For each positive integer n let A_n be the closed annulus consisting of all points x such that $n-1 \leq |x| \leq n$. Then for each $y \in \mathbf{R}^n$ the set $N_{1/2}(y)$ only contains points from at most two closed sets in the family (verify this!).

(2) For each positive integer n let V_n be the open annulus consisting of all points x such that $n-2 < |x| < n+1$. The details for this example are left to the reader as an exercise.

Locally finite families are useful in many contexts. For example, we have the following result:

PROPOSITION. *If X is a topological space and $\mathcal{A} = \{ A_\alpha \}$ is a locally finite family of closed subsets (not necessarily finite), then $\cup_\alpha A_\alpha$ is also closed.*

Proofs of this and other basic results on locally finite families of subsets appear on pages 112 and 244–245 of Munkres.■

Here is the ultimate result on metrization.

NAGATA-SMIRNOV METRIZATION THEOREM. *A topological space is metrizable if and only if it is* $\mathbf{T_3}$ *and there is a base that is a countable union of locally finite families (also known as a σ-locally finite base).*

This is Theorem 40.3 on page 250 of Munkres, and the a proof including preliminary observations is contained in Section 40 on pages 248–252. Note that the Urysohn Metrization Theorem is an immediate consequence of this result because a countable base is a countable union of families such that each has exactly one element.■

A somewhat different metrization theorem due to Smirnov is in some sense the ultimate result on finding a metric on spaces built out of metrizable pieces: *A Hausdorff space is metrizable if it is paracompact and locally metrizable.* The converse to this result is also true and is due to A. H. Stone (metrizable $\Longrightarrow$ paracompact). Further information on these results appears in the files `smirnov.*` in the course directory.