

# Category Theory

Category theory is a general mathematical theory of structures, systems of structures and relationships between systems of structures. It provides a unifying and economic mathematical modeling language. Category theory lends itself very well to extracting and generalizing elementary and essential notions and constructions from many mathematical disciplines (and thus it is half-jokingly known as “abstract nonsense”). Thanks to its general nature, the language of category theory enables one to “transport” problems from one area of mathematics, via suitable transformations known as “functors,” to another area, where the solution may be easier to find; there is a clear parallel with the Laplace transform, which allows one to find solutions to differential equations by translating an analytic problem into a purely algebraic question. Although category theory was originally developed to deal more precisely with relationships between topology and algebra, it soon became more than a convenient language and evolved into a field in itself. It is currently used in a wide range of areas within mathematics as well as some areas of computer science and mathematical physics. In fact, some mathematicians have used category theory to construct an alternative to set theory as a foundation for mathematics.

## Categories

A **category** attempts to capture the essence of a class of related mathematical objects, for instance the class of groups. Instead of focusing on the individual objects (groups) in the traditional manner, the *morphisms* — *i.e.*, the structure preserving maps — between these objects receive almost equal emphasis. In the example of groups, these are the group homomorphisms.

**Definition.** A category  $\mathcal{C}$  is a triple consisting of the following pieces of data:

- a class (or collection)  $\text{Obj}(\mathcal{C})$  of things called the objects of the category.
- for every two objects  $A$  and  $B$  a set  $\text{Mor}(A, B)$  of things called morphisms from  $A$  to  $B$ . If  $f$  is in  $\text{Mor}(A, B)$ , we write  $f : A \rightarrow B$ .
- for every three objects  $A, B$  and  $C$  a binary operation  $\text{Mor}(B, C) \times \text{Mor}(A, B) \rightarrow \text{Mor}(A, C)$  called composition of morphisms. The composition of the two morphisms  $g : B \rightarrow C$  and  $f : A \rightarrow B$  is written as  $g \circ f$  or simply  $gf$ .

These are assumed to satisfy the following conditions:

- The various sets  $\text{Mor}(A, B)$  are pairwise disjoint (hence the domain  $A$  and the codomain  $B$  of a morphism are well defined).

- (*associativity*) If  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h : C \rightarrow D$  then we have  $h \circ (g \circ f) = (h \circ g) \circ f$ .
- (*identity*) For every object  $X$ , there exists a unique morphism  $\mathbf{1}_X : X \rightarrow X$  called the *identity morphism for X*, such that for every morphism  $f : A \rightarrow B$  we have  $\mathbf{1}_B \circ f = f = f \circ \mathbf{1}_A$ .

If the class of objects lies inside some set, the category is said to be *small*. The two most trivial examples of (small) categories are the category  $\square$  which has only one object and one morphism (the identity map) and the category  $\square\square$  which has two objects  $A, B$  and morphisms given by the identities of  $A$  and  $B$  plus an additional morphism  $f : A \rightarrow B$ . Many important categories are not small, but many others are (see also the definition of *locally small* categories below).

## Examples

One of the interesting features of category theory is that it provides a uniform treatment of the notion of structure. This can be seen, first, by considering the variety of examples of categories. Almost every known example of a mathematical structure with the appropriate structure preserving map yields a category. However, as indicated by the list below, there are also other examples of categories; as always in mathematics, any entity satisfying the conditions given in the definition is a category.

Each category below is presented in terms of its objects and its morphisms.

- The category **Grp** consisting of all groups together with their group homomorphisms.
- The category **Ab** consisting of all abelian groups together with their group homomorphisms.
- The category **Ring** consisting of all (associative) rings together with their ring homomorphisms.
- The category **Vect( $F$ )** of all vector spaces over the field  $F$  together with their  $F$  – linear transformations.
- The category **Set** of all sets together with functions between sets.
- The category **Op-Int** of all open intervals in the real line with continuous functions.
- Any *partially ordered set*  $(P, \leq)$  forms a category, where the objects are the members of  $P$ , and the morphisms are arrows pointing from  $x$  to  $y$  precisely when  $x \leq y$ .
- Any *monoid*  $M$  (*i.e.*, a set with an associated binary operation and two sided identity) forms a small category with a single object  $M$ , and where every element of the monoid is a morphism from  $M$  to  $M$  (the monoid operation yields the categorical composition of morphisms).
- Any directed graph can be considered as a small category: the objects are the vertices of the graph and the morphisms are the paths in the graph. Composition of morphisms is

concatenation of paths (*i.e.*, join them together at the final point of the first path, which is merely the same as the first point of the second path).

- If  $\mathcal{S}$  is a set, the **discrete category on  $\mathcal{S}$**  is the small category which has the elements of  $\mathcal{S}$  as objects and only the identity morphisms as morphisms.
- Any category  $\mathcal{C}$  can itself be considered as a new category in a different way: The objects are the same as those in the original category but the arrows are those of the original category reversed. This is called the **dual** or **opposite category** and is denoted by  $\mathcal{C}^{\text{OP}}$ .
- If  $\mathcal{C}$  and  $\mathcal{D}$  are categories, one can form the **product category**  $\mathcal{C} \times \mathcal{D}$ : The objects are pairs consisting of one object from  $\mathcal{C}$  and one from  $\mathcal{D}$ , and the morphisms are also pairs, consisting of one morphism in  $\mathcal{C}$  and one in  $\mathcal{D}$ . Such pairs can be composed componentwise.

## Special types of morphisms

Let  $X$  be an arbitrary object in  $\mathcal{C}$ . A morphism  $f : A \rightarrow B$  in  $\mathcal{C}$  is called a/an

- **monomorphism** if  $f g_1 = f g_2$  implies  $g_1 = g_2$  for all morphisms  $g_1, g_2 : X \rightarrow A$ ,
- **epimorphism** if  $g_1 f = g_2 f$  implies  $g_1 = g_2$  for all morphisms  $g_1, g_2 : B \rightarrow X$ ,
- **isomorphism** if there exists a morphism  $g : B \rightarrow A$  with  $f g = \mathbf{1}_B$  and  $g f = \mathbf{1}_A$ ,
- **retract** if there exists a morphism  $g : B \rightarrow A$  with  $g f = \mathbf{1}_A$ ,
- **retraction** if there exists a morphism  $g : B \rightarrow A$  with  $f g = \mathbf{1}_B$ ,
- **automorphism** if  $f$  is an isomorphism and  $A = B$ .

In many standard categories such as the category of sets and functions or the categories of vector spaces and linear transformations, monomorphisms turn out to be maps that are one-to-one and epimorphisms turn out to be maps that are onto (check this out on your own!). Furthermore, in these categories one also has that the retracts and monomorphisms are the same, and similarly the retractions and epimorphisms are the same. In an arbitrary category one knows that retracts and retractions are monomorphisms and epimorphisms respectively (verify this!), but the converse statements do are false in general (for example, the inclusion of the integers mod 2 in the integers mod 4 sending 1 to 2 is a monomorphism of abelian groups that is not a retract, and the projection from the integers mod 4 to the integers mod 2 is an epimorphism of abelian groups that is not a retraction).

There are contexts in which it is useful to consider categories in which every map is an isomorphism. Such categories are known as **groupoids**.

Having defined isomorphisms we can now formulate an extremely useful generalization of the concept of a small category. Before doing so we need a few preliminary definitions.

**Definition.** Let  $\mathcal{C}$  be a category; a subcategory of  $\mathcal{C}$  is a category  $\mathcal{S}$  such that

- (1) every object or morphism in  $\mathcal{S}$  is also (respectively) an object or morphism of  $\mathcal{C}$ ,
- (2) if a morphism of  $\mathcal{C}$  lies in  $\mathcal{S}$ , then so do its domain and codomain as well as their identity maps,
- (3) if the morphisms  $f$  and  $g$  lie in  $\mathcal{S}$  and their composite in  $\mathcal{C}$  is defined, then this composite also lies in  $\mathcal{S}$ .

**Definition.** A subcategory  $\mathcal{S}$  of  $\mathcal{C}$  is said to be a full subcategory if for all objects  $A$  and  $B$  in  $\mathcal{S}$  the  $\mathcal{S}$ -morphisms from  $A$  to  $B$  are the same as the  $\mathcal{C}$ -morphisms from  $A$  to  $B$ .

In particular, the category of abelian groups is a full subcategory of the category of all groups.

**Definition.** A category  $\mathcal{C}$  is said to be locally small if there is a small full subcategory  $\mathcal{S}$  such that every object in  $\mathcal{C}$  is isomorphic to an object in  $\mathcal{S}$ .

If  $F$  is a field, then the category  $\mathcal{C}$  of finite-dimensional vector spaces over  $F$  (with linear transformations as the morphisms) is not a small category, but if  $\mathcal{S}$  is the full subcategory whose objects are the coordinate spaces  $F^n$ , then every object in  $\mathcal{C}$  is isomorphic to an object in  $\mathcal{S}$  and thus  $\mathcal{C}$  is locally small.

It is important to note that the morphisms are the crucial structure that characterizes a category rather than objects. Thus, a category of vector spaces over the real numbers with *additive* maps as the morphisms is different from the category of vector spaces over the real numbers with linear transformations as the morphisms. In particular, the latter will have different properties as a category when compared to the former (for example, all nonzero finite dimensional vector spaces are isomorphic in the former but not in the latter!). The previous examples have something in common: the objects are all structured sets with structure preserving maps. However, any entity satisfying the conditions given in the definition is a category.

## Functors

An important underlying idea in category theory is that when one defines a type of mathematical object, there should be a corresponding definition of morphisms between objects of the given type. Needless to say, this should apply to categories themselves, and **functors** are the structure-preserving maps between categories. Many of the “natural” or “canonical” constructions in mathematics can be expressed as functors.

**Definition.** A (covariant) functor  $F$  from one category  $\mathcal{C}$  to another category  $\mathcal{D}$  assigns

- to each object  $X$  in  $\mathcal{C}$  an object  $F(X)$  in  $\mathcal{D}$ ,
- to each morphism  $f : X \rightarrow Y$  a morphism  $F(f) : F(X) \rightarrow F(Y)$

such that the following two properties hold:

- $F(1_X) = 1_{F(X)}$  for every object  $X$  in  $\mathcal{C}$ ,
- $F(g \circ f) = F(g) \circ F(f)$  for all morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ .

A contravariant functor  $F$  between categories  $\mathcal{C}$  and  $\mathcal{D}$  is a functor that “turns morphisms around.” The quickest way to define it is as a covariant functor between  $\mathcal{C}^{\text{OP}}$  and  $\mathcal{D}$ , but due to the importance of the concept we shall write out the definition more explicitly:

**Definition.** A contravariant functor  $F$  from one category  $\mathcal{C}$  to another category  $\mathcal{D}$  assigns

- to each object  $X$  in  $\mathcal{C}$  an object  $F(X)$  in  $\mathcal{D}$ ,
- to each morphism  $f : X \rightarrow Y$  a morphism  $F(f) : F(Y) \rightarrow F(X)$

such that the following two properties hold:

- $F(1_X) = 1_{F(X)}$  for every object  $X$  in  $\mathcal{C}$ ,
- $F(g \circ f) = F(f) \circ F(g)$  for all morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ .

## Examples of Functors

**Sets to vector spaces:** An example of a *covariant* functor from the category of sets to the category of all real vector spaces may be defined as follows. Given a set  $A$  let  $V(A)$  be the “real vector space which has  $A$  as a basis.” Formally, an element of  $V(A)$  is a real valued function  $\mathbf{v}$  on  $A$  such that  $\mathbf{v}(\mathbf{a}) = 0$  for all but finitely many  $\mathbf{a} \in A$ , with addition and real multiplication defined in the usual way. Following standard practice, we shall write a typical element of  $V(A)$  as a sum

$$\sum t_a \cdot a$$

where  $t_a = 0$  for all but finitely many  $\mathbf{a}$  and the elements of  $\mathbf{a}$  correspond to the basis elements

(i.e., the functions that are 1 at the element  $a$  of  $A$  and 0 at all others). If  $f : A \rightarrow B$  is a set-theoretic function then there is a unique linear transformation  $V(f) : V(A) \rightarrow V(B)$  that sends  $s$  to  $f(s)$ . It is left as an exercise to verify that this construction defines a covariant functor.

**Dual vector space:** A less trivial example of a *contravariant* functor from the category of all real vector spaces to itself is given by assigning to every vector space its dual space and to every linear map its dual or transpose; specifically, if  $f$  is a linear functional on  $V$  and  $T : W \rightarrow V$  is a linear transformation, then  $T^* : W^* \rightarrow V^*$  is the linear transformation sending  $f$  to  $f \circ T$ .

**Homomorphism groups:** To every pair  $A, B$  of abelian groups one can assign the abelian group  $\mathbf{Hom}(A, B)$  consisting of all group homomorphisms from  $A$  to  $B$ . This is a functor which is contravariant in the first and covariant in the second argument; i.e., it is a covariant functor

$$\mathbf{Ab}^{\text{OP}} \times \mathbf{Ab} \rightarrow \mathbf{Ab}$$

(where  $\mathbf{Ab}$  denotes the category of abelian groups with group homomorphisms as the morphisms). If both  $f : A_1 \rightarrow A_2$  and  $g : B_1 \rightarrow B_2$  are morphisms in  $\mathbf{Ab}$ , then the group homomorphism

$$\mathbf{Hom}(f, g) : \mathbf{Hom}(A_2, B_1) \rightarrow \mathbf{Hom}(A_1, B_2)$$

is given by  $\square \alpha g \circ \square \circ f$ . If one holds the first variable (i.e., the abelian group  $A$ ) fixed, then one obtains a covariant functor

$$\mathbf{Hom}(A, -) : \mathbf{Ab} \rightarrow \mathbf{Ab}$$

(specifically, on objects it sends the group  $G$  to  $\mathbf{Hom}(A, G)$  and on morphisms it sends  $h$  to  $\mathbf{Hom}(1_A, h)$ ) and if one holds the second variable (i.e., the abelian group  $B$ ) fixed, then one obtains a contravariant functor

$$\mathbf{Hom}(-, B) : \mathbf{Ab} \rightarrow \mathbf{Ab}$$

(specifically, on objects it sends the group  $G$  to  $\mathbf{Hom}(G, B)$  and on morphisms it sends  $h$  to  $\mathbf{Hom}(h, 1_B)$ ).

**Vector spaces of linear transformations:** Similar considerations hold for vector spaces over a field  $F$  and  $F$ -linear transformations between them; in this case the set  $\mathbf{Lin}(A, B)$  of all linear transformations from  $A$  to  $B$  is a vector space over  $F$ .

**Forgetful functors:** Consider the functor  $F : \mathbf{Ring} \rightarrow \mathbf{Ab}$  which maps a ring to its underlying abelian additive group. Morphisms in  $\mathbf{Ring}$  (ring homomorphisms) become morphisms in  $\mathbf{Ab}$  (abelian group homomorphisms).

**Power set functors:** Given a set  $X$ , let  $\mathcal{P}[X]$  denote the set of all subsets of  $X$  (*i.e.*, the power set of  $X$ ). This construction leads to a pair of functors on the category of sets, one of which is covariant and the other of which is contravariant. Specifically, if  $f : X \rightarrow Y$  is a map of sets define

$$\mathcal{P}_* [f] : \mathcal{P}[X] \rightarrow \mathcal{P}[Y] \quad \text{by the formula} \quad \mathcal{P}_* [f](A) = f(A) \subset Y$$

for  $A \subset X$ , and similarly define

$$\mathcal{P}^* [f] : \mathcal{P}[X] \rightarrow \mathcal{P}[Y] \quad \text{by the formula} \quad \mathcal{P}^* [f](B) = f^{-1}(B) \subset X$$

for  $B \subset Y$ . It is easily verified that these define functors (see the exercises in Munkres and the remarks in the earlier notes on set theory).

**Remark.** If  $S$  is a set, then  $\mathcal{P}[S]$  has an algebraic structure given by unions and intersections, and these operations make it into a type of system known as a Boolean algebra.

The exercises in Munkres show that  $\mathcal{P}^* [f]$  is always a homomorphism of Boolean algebras (because taking inverse images preserves unions and intersections), and thus can be viewed as a contravariant functor from the category of sets to a suitably defined category of Boolean algebras. In contrast, the map  $\mathcal{P}_* [f]$  is not necessarily a homomorphism of Boolean algebras because it does not necessarily preserve intersections or the unit element, but it does define a homomorphism of the monoid structure determined by set-theoretic union (where the empty set is the identity element), so it can be viewed as a covariant functor from the category of sets to a suitably defined category of monoids.

## Categorical Constructions

Category theory unifies mathematical structures in another, perhaps more far-reaching, manner. Once a type of structure has been defined, it quickly becomes imperative to determine how new structures can be constructed out of the given one and how given structures can be decomposed into more elementary substructures. For instance, given two sets  $A$  and  $B$ , set theory allows us to construct their cartesian product  $A \times B$ . For an example of the second sort, given a finite abelian group, it can be decomposed into a product of some of its subgroups. In both cases, it is necessary to know how structures of a certain kind combine. The nature of these combinations might appear to be considerably different when looked at from too close. Category theory reveals

that many of these constructions are in fact special cases of objects in a category with what is called a "universal property". Indeed, from a categorical point of view, a set-theoretical cartesian product, a direct product of groups, a direct product of abelian groups, and numerous other such constructions are all instances of a categorical concept: the categorical product. What characterizes the latter is a universal property. Formally, a product for two objects  $A$  and  $B$  in a category  $\mathcal{C}$  is an object  $P$  of  $\mathcal{C}$  together with two morphisms called the projections

$$u : P \rightarrow A \quad \text{and} \quad v : P \rightarrow B$$

such that the following universal property holds:

*For all objects  $X$  with pairs of morphisms  $f : X \rightarrow A$  and  $g : X \rightarrow B$ , there is a unique morphism  $h : X \rightarrow P$  such that  $u \circ h = f$  and  $v \circ h = g$ .*

Notice that we have defined A product for  $A$  and  $B$  and not THE product for  $A$  and  $B$ . Indeed, products, and in fact all objects with universal properties are defined up to (a unique) isomorphism. Thus, in category theory the actual process for carrying out a certain construction is irrelevant, and the important point is the way an object is related to the other objects of the category by the morphisms going in and the morphisms going out; *i.e.*, how certain structures can be mapped into or out of it and how it can map its structure into other structures of the same kind.

## Natural transformations

Two functorial constructions are often "naturally related" in some sense, and this leads to the concept of *natural transformation*, which may be viewed as a "morphism of functors."

**Definition.** If  $F$  and  $G$  are (covariant) functors from the category  $\mathcal{C}$  to the category  $\mathcal{D}$ , then a natural transformation from  $F$  to  $G$  associates to every object  $X$  in  $\mathcal{C}$  a morphism  $\eta_X : F(X) \rightarrow G(X)$  in  $\mathcal{D}$  such that for every morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  we have

$$\eta_Y \circ F(f) = G(f) \circ \eta_X.$$

The two functors  $F$  and  $G$  are said to be naturally isomorphic if there exists a natural transformation from  $F$  to  $G$  such that  $\eta_X$  is an isomorphism for every object  $X$  in  $\mathcal{C}$ .

**Modification for contravariant functors.** If  $F$  and  $G$  are contravariant functors from the category  $\mathcal{C}$  to the category  $\mathcal{D}$ , there is a similar definition of natural transformation, the only difference being that the final equation is replaced by  $\eta_X \circ F(f) = G(f) \circ \eta_Y$ .



The need to understand natural equivalence was the basic motivation for S. Eilenberg and S. MacLane to develop the notions of category theory. Their original objective was to make the notion of natural equivalence mathematically precise, and of course this was formulated in terms of natural transformations. In order to give a general definition of the latter, they defined the notion of functor, borrowing the terminology from Rudolf Carnap (a twentieth century philosopher whose work dealt extensively with the philosophy of science and semantics), and in order to give a general definition of functor, they defined the notion of category, borrowing this time from Kant and Aristotle.

## Examples

**Double dual spaces:** If  $F$  is a field, then for every vector space  $V$  over  $F$  we have a “natural” linear evaluation map  $e_V: V \rightarrow V^{**}$  (= the double dual space of  $V$ ) sending  $x \in V$  to the linear functional on  $V^*$  given by evaluation at  $x$ , and this is an isomorphism of vector spaces when  $V$  is finite-dimensional. These maps are “natural” in the following sense: The double dual operation is a covariant functor, and the linear transformations determine a natural transformation from the identity functor to the double dual space functor. In contrast, if  $V$  is finite-dimensional then  $V$  is isomorphic to its own dual space  $V^*$ , but in order to specify an isomorphism one needs some extra “unnatural” structure like an ordered basis or, over the real numbers, an inner product.

**Power set functors:** Let  $S$  be a set, and let  $\mathcal{P}[-]$  be the covariant power set functor described earlier. Consider the function

$$\theta_S : S \rightarrow \mathcal{P}[S]$$

sending  $x \in S$  to the singleton set  $\{x\}$ , which is a subset of  $S$  and thus an element of  $\mathcal{P}[S]$ . The system of such functions can be viewed as a natural transformation

$$\theta : \mathbf{I} \rightarrow \mathcal{P}$$

where  $\mathbf{I}$  denotes the identity functor on the category of sets, because of the commutativity condition

$$\theta_Y \circ \mathbf{I}(f) = \mathcal{P}[f] \circ \theta_X$$

which holds because the values of both sides at an arbitrary  $x \in S$  are equal to  $\{f(x)\} \subset Y$ .

## Equivalence of categories

Intuitively, two categories are equivalent if they cannot be distinguished from the standpoint of category theory.

**Definition.** Two categories  $\mathcal{C}$  and  $\mathcal{D}$  are said to be equivalent if there exist covariant functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that  $FG$  is naturally isomorphic to  $1_{\mathcal{D}}$  (where  $1_{\mathcal{D}}$  denotes the identity functor  $\mathcal{D} \rightarrow \mathcal{D}$  which assigns every object to itself and every morphism to itself) and  $GF$  is naturally isomorphic to  $1_{\mathcal{C}}$ .

## Final Remarks

The discussion above summarizes the elementary concepts of category theory, but it does not get into the most important and powerful notions of the subject, including the notion of adjoint functors, which turns out to play a key role in many mathematical contexts. Such material goes well beyond what is needed for first year graduate courses.