Addendum to Section III.3 : Finding roots of cubic polynomials

This note works out the details of some comments in the lectures about using the Contraction Lemma to find roots of cubic polynomials that cannot be factored over the rational numbers.

Consider the polynomial $p(x) = x^3 - x - 1$. We claim this is irreducible over the rationals, which by a result of Gauss is equivalent to irreducibility over the integers. If it factored nontrivially over the integers, it would necessarily have a linear factor (if $m + n = 3$ for positive integers m and n, one of them must be equal to 1). Furthermore, if $x - a$ were such a linear factor, then a would have to equal ± 1. Since one can easily ncheck directly that $x \pm 1$ is not a factor of $p(x)$, it follows that the latter is irreducible. As an odd degree polynomial it must have at least one real root.

Since $p(1) < 0 < p(2)$ it follows that there is a real root of p in the interval $(1, 2)$. We shall evaluate this root using the Contraction Lemma.

In order to apply the Contraction Lemma it is necessary to find a function T such that $p(x) = 0$ if and only if $T(x) = x$. One natural choice for T in this problem would be $T(x) = x^3 - 1$. However, one encounters difficulties with this choice because it is necessary to choose T so that the following additional properties hold:

(1) There is an interval $[a, b]$ containing the root r such that T maps $[a, b]$ into itself.

(2) The map T satisfies the condition $|T(u) - T(v)| \leq \alpha\,|u - v|$ for some $\alpha \in (0, 1)$.

The second condition will follow from the Mean Value Theorem if T is differentiable on (a, b) and satisfies $|T'(x)| \leq \alpha$ for all $x \in (a, b)$.

If one tries to check these conditions for the choice of T suggested above, one encounters difficulties. These can be avoided by finding anonther candidate for T; specifically, $T(x) = \sqrt[3]{x + 1}$.

In this case it is easy to check both of the conditions. First note that

$$\frac{dT}{dx} = \frac{1}{3\,\sqrt[3]{(x+1)^2}}$$

and consequently T is increasing on $[1, 2]$ because its derivative is positive. Next, note that $T(1) = \sqrt[3]{2}$ and $T(2) = \sqrt[3]{3}$ where each of these cube roots is greater than 1 because $1^3 = 1$ and also less than 2 because $2^3 = 8$. Therefore Condition (1) holds in this case. What about the condition on the derivative? The latter is a decreasing positive valued function of x on the interval $[1, 2]$, and therefore it is only necessary to check that $T'(1) < 1$. But the formula above implies that

$$T'(1) = \frac{1}{3\,\sqrt[3]{2}}$$

which is less than $1/3$, and consequently $|T'(x)| \leq 1/3$ for $x \in (1, 2)$.

Therefore the real root of $p(x)$ on the interval $(1, 2)$ is the limit of the sequence of points x_n where $x_0 = 1$ and $x_{n+1} = T(x_n)$. Here is what happens if one carries out the computation of the sequence on a scientific calculator:

1.0000000
1.2599181
1.3122897
1.3223493

1.3242642
1.3246280
1.3246972
1.3247103

1.3247128
1.3247132
1.3247134
1.3247135

1.3247136
1.3247136
etc.

The final line indicates that one will get the same answer for all subsequent iterations; therefore the next to last line gives the real root of $p(x)$ in $(1,2)$ to seven decimal places.

Extracting cube roots

One can also construct an algorithm for finding the cube root of a real number greater than 1 that is parallel to the algorithm for square roots described in one of the exercises from Edwards. Specifically, if $a > 1$ is the number whose cube root is to be found, then the formula for an appropriate contraction operator is

$$T(x) = \frac{1}{3} \cdot \left(2\,x + \frac{a}{x^2} \right) .$$

As in the case of square roots, one starts with some real number y such that $y > a^3$ and checks to see whether the fundamental conditions (1) and (2) both hold for T with respect to the inverval

$$\left[\sqrt[3]{a}, y\right] .$$

To see that T does map the latter interval into itself it suffices to verify that $T(y)$ itself lies in the given interval (if z lies in the corresponding open interval, the same argument will imply that $T(z)$ lies in $[\sqrt[3]{a}, z]$). But this is true because

$$T(y) \quad = \quad \frac{1}{3} \cdot \left(2\,y + \frac{a}{y^2} \right) \quad < \quad \frac{1}{3} \cdot \left(2\,y + \frac{y^3}{y^2} \right) \quad = \quad y \; .$$

To verify the required condition on the derivative, first note that

$$\frac{dT}{dx} = \frac{2}{3} \cdot \left(1 \; - \; \frac{a}{x^3} \right)$$

Since $x^3 \geq a$ it follows that the derivative at x lies in the closed interval $[0, \frac{2}{3}]$, and therefore we have verified that the conditions of the Contraction Lemma apply.■