More metrics on cartesian products

If $(X_i, \mathbf{d}_i)$ are metric spaces for $1 \leq i \leq n$, then in Section II.4 of the lecture notes we defined three metrics on $\prod_i X_i$ whose underlying topologies are the product topology. The purpose of this note is to explain how one can interpolate a continuous family of metrics between these examples; for each such metric, the underlying topology will be the product topology.

Throughout this discussion $p \geq 1$ will denote a fixed real number.

Let $x, y \in \prod_i X_i$, express them in terms of coordinates as $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ respectively, and define $\mathbf{d}_p$ from $\prod_i X_i \times \prod_i X_i$ to $\mathbf{R}$ as follows:

$$\mathbf{d}^{\langle p \rangle}(x,y) \quad = \quad \left(\sum_i \mathbf{d}_i(x_i, y_i)^p \right)^{1/p}$$

The cases where $p = 1$ or 2 were considered in the lecture notes.

It follows immediately that $\mathbf{d}^{\langle p \rangle}$ satisfies all the properties for a metric except perhaps the fundamentally important Triangle Inequality. The latter is in fact a consequence of the following basic result:

Minkowski's Inequality. *Suppose that we have $u, v \in \mathbf{R}^n$ and we write these vectors in coordinates as $(u_1, \cdots, u_n)$ and $(v_1, \cdots, v_n)$ respectively. Then we have*

$$\left(\sum_i |u_i + v_i|^p \right)^{1/p} \quad \leq \quad \left(\sum_i |u_i|^p \right)^{1/p} \; + \; \left(\sum_i |v_i|^p \right)^{1/p} \quad . \blacksquare$$

Here are some references for a proof of Minkowski's Inequality:

W. Rudin, Real and Complex Analysis. (Third Edition. Mc-Graw-Hill Series in Higher Mathematics.) *McGraw-Hill, Boston*-etc., 1987. ISBN: 0-07-054234-1.

`http://www.planetmath.org/encyclopedia/MikowskiInequality.html`

The incorrect spelling "Mikowski" needed to reach the `planetmath` link should be noted; the latter also gives further links to the closely related Hölder Inequality; in fact, one geneerally begins by proving Hölder's Inequality and then derives Minkowski's inequality from Hölder's Inequality.

Hölder's Inequality. *Suppose that we have $u, v \in \mathbf{R}^n$ as above with $p > 1$, and that we choose $q > 1$ such that*

$$\frac{1}{q} + \frac{1}{p} \quad = \quad 1 \; .$$

Then we have

$$\left(\sum_i |u_i \cdot v_i| \right) \quad \leq \quad \left(\sum_i |u_i|^p \right)^{1/p} \cdot \left(\sum_i |v_i|^q \right)^{1/q} \quad . \blacksquare$$

Since each of the metrics $\mathbf{d}^{\langle p \rangle}$ for $p = 1, 2, \infty$ defines the product topology, it is natural to speculate that the same holds for all choices of p, and in fact this is true.

PROPOSITION. *For each $p \geq 1$, the topology determined by the metric $\mathbf{d}^{\langle p \rangle}$ is the product topology. Furthermore, the identity map from $(\prod_i X_i, \mathbf{d}^{\langle \alpha \rangle})$ to $(\prod_i X_i, \mathbf{d}^{\langle \beta \rangle})$ is uniformly continuous for all choices of α, β such that $1 \leq \alpha, \beta \leq \infty$.*

Proof. It suffices to prove the assertion in the second sentence, and the latter reduces to the special case where one of α, β is ∞; if we know the result in such cases, we can retrieve the general case using the uniform continuity of the identity mappings

$$\left(\prod_i X_i, \mathbf{d}^{\langle p \rangle}\right) \longrightarrow \left(\prod_i X_i, \mathbf{d}^{\langle \infty \rangle}\right) \longrightarrow \left(\prod_i X_i, \mathbf{d}^{\langle r \rangle}\right)$$

and the fact that a composite of uniformly continuous maps is uniformly continuous.

The uniform continuity statements are direct consequences of the following inequalities for nonnegative real numbers $u`_i$ for $1 \leq i \leq n$:

$$\max_i \ \{ \ u_i \ \} \ \leq \ \left(\sum_i \ u_i^p\right)^{1/p} \ \leq \ n \cdot \max_i \ \{ \ u_i \ \}$$

One can then apply the argument in the notes to show that the identity maps

$$\left(\prod_i X_i, \mathbf{d}^{\langle \infty \rangle}\right) \to \left(\prod_i X_i, \mathbf{d}^{\langle p \rangle}\right) \to \left(\prod_i X_i, \mathbf{d}^{\langle \infty \rangle}\right)$$

are uniformly continuous (and in fact the δ corresponding to a given ε can be read off explicitly from the inequalities!), and of course all composites of maps from this diagram are also uniformly continuous.■

The limiting case

The following result is the motivation for setting $\mathbf{d}_\infty$ equal to the maximum distance between coordinates:

PROPOSITION. *In the setting above we have*

$$\mathbf{d}_\infty \ = \ \lim_{p \to \infty} \ \mathbf{d}_p \ .$$

Proof. This reduces immediately to proving the following result: *If* $u \in \mathbf{R}^n$ *as above then*

$$\max_i \ \{|u_i|\} \ = \ \lim_{p \to \infty} \ \left(\sum_i \ |u_i|^p\right)^{1/p} \ .$$

Let M denote the expression on the left hand side, and for each $p > 1$ let Y_p denote the value of the expression whose limit we wish to find. Clearly $M \leq Y_p$ for all p because M is obtained by deleting all but one summand from Y_p. However, since $|u_i| \leq M$ for all i, we also have $Y_p \leq (n \cdot M^p)^{1/p} = M \cdot n^{1/p}$. Now the limit of the right hand side as $p \to \infty$ is equal to M, and thus we have sandwiched Y_p between two expressions, one of which is equal to M and the other of which has a limit equal to M. It follows that the limit of Y_p is also equal to M, which is exactly the claim in the proposition.■

If one graphs the set of all points in $\mathbf{r}^2$ whose p-diesance from the origin is equal to 1 for varoius values of $p \geq 1$, the result is a collection of closed curves centered at the origin such that the area enclosed by the curve increases with p and the limit of these curves is the square whose vertices are the elements of the set $\{\pm 1\} \times \{\pm 1\}$.