NAME:____

Mathematics 205A, Fall 2005, Examination 1

Point values are indicated in brackets.

1. [*25 points*] Let X be a topological space, let $B \subset X$, and suppose that for each $b \in B$ there is an open set U_b such that $b \in U_b \subset B$. Prove that B is open in X.

SOLUTION.

For the sake of completeness we note that each U_b is assumed to be open in X.

We have $U_b \subset B$ for all $b \in B$, and hence

$$B \quad = \quad \bigcup_{b \in B} \{b\} \quad \subset \quad \bigcup_{b \in B} U_b \quad \subset \quad B$$

which in turn implies that we have equality between each pair of sets in the displayed line. Since each U_b is open in X, it follows that $B = \cup_b \, U_b$ is also open in X.■

2. [*25 points*] Let X be a set and let $\mathbf{d} : X \times X to \mathbf{R}$ be a function which satisfies the conditions (1) $\mathbf{d}(x,y) \geq 0$ with equality if and only if $x = y$ and (2) $\mathbf{d}(x,y) = \mathbf{d}(y,x)$ for all $x, y \in X$. One says that $(X, \mathbf{d})$ is an *ultrametric space* if $\mathbf{d}$ also satisfies the condition $\mathbf{d}(x,y) \leq \min\big(\mathbf{d}(x,z), \mathbf{d}(y,z)\big)$ for all $x, y, z \in X$.

[*i*] Prove that an ultrametric space satisfies the triangle inequality.

[*ii*] Is $\mathbf{R}$ with the usual metric an ultrametric space? Prove this or find x, y, z such that the defining inequality fails.

SOLUTION.

First part. This follows because if $u, v \geq 0$ then $\min\{u, v\} \leq u + v$.■

Sexcond part. The real line is not ultrametric because if $x = 0$, $y = 2$ and $z = 1$, then $\mathbf{d}(x, 2) = 2$ but $\mathbf{d}(x,z) = \mathbf{d}(z,y) = 1$, so that $\mathbf{d}(x,z) = 2 > 1 = \mathbf{d}(x,z) = \mathbf{d}(z,y)$.■

IMPORTANT COMMENT.

The usual definition of an ultrametric space has the contrasting condition $\mathbf{d}(x,y) \leq \max\big(\mathbf{d}(x,z), \mathbf{d}(y,z)\big)$, but the answer to the problem is basically the same whether one uses maxima or minima. In particular, the first statement is true because if $u, v \geq 0$ then $\max\{u, v\} \leq u + v$, and the given example works regardless of whether one has maxima or minima. — On the other hand, there are many examples of metric spaces satisfying $\mathbf{d}(x,y) \leq \max\big(\mathbf{d}(x,z), \mathbf{d}(y,z)\big)$, but the only examples for which $\mathbf{d}(x,y) \leq \min\big(\mathbf{d}(x,z), \mathbf{d}(y,z)\big)$ consist of one point sets. To see this, note that if we take $y = z$ then the latter reduces to $\mathbf{d}(x,y) \leq \min\big(\mathbf{d}(x,y), \mathbf{d}(y,y)\big) = \min\big(0, \mathbf{d}(x,y)\big) = 0$. But this means that $x = y$, and since x and y were arbitrary, then X must consist of rxactly one point.■

3. [*25 points*] Let X be a topological space, and let $T : X \times X \times X \to X \times X \times X$ be the map sending (x, y, z) to (z, x, y). Prove that T is a homeomorphism. [*Hints:* Consider the projections of T onto the three coordinates, and also look at the composites $T^2 = T \circ T$, $T^3 = T^2 \circ T = T \circ T^2$, and so on.]

SOLUTION.

We first show that T is continuous by considering its projections p_X, p_Y, p_Z onto X, Y, Z respectively. But $p_X \circ T = p_Z$, $p_Y \circ T = p_X$, and $p_Z \circ T = p_Y$, so T is continuous because its projections are. To see that T is a homeomorphism, note that T^3 is the identity, and therefore the equations $T \circ T^2 = T^3 = \mathrm{id} = T^3 = T^2 \circ T$ imply that T is bijective and the inverse of T is T^2, which is continuous because T is continuous.∎

4. [*25 points*] Let X be a topological space, let $A \subset X$, let $\mathbf{L}(A)$ be the set of limit points of A, and suppose that A and $\mathbf{L}(A)$ are disjoint. Prove that the subspace topology on A is equal to the discrete topology. [*Hint:* If $a \in A$, why do we know that $a \notin \mathbf{L}(A)$?]

SOLUTION.

If $a \in A$, then $a \notin \mathbf{L}(A)$ because $A \cap \mathbf{L}(A) = \emptyset$. Since A is not a limit point of A it follows that there is some open neighborhood U_a of a in X such that $U - \{a\} \cap A = \emptyset$. The latter means that $U \cap A$, which contains a by assumption, is equal to $\{a\}$. Therefore $\{a\}$ is open in the subspace topology for A. Since A is arbitrary, this means that every one point subset of A is open in the subspace topology and hence every subset is open in this topology, so that the latter must be discrete.∎