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Mathematics 205A, Fall 2005, Examination 2

Point values are indicated in brackets.

1. [*20 points*] (*i*) If X is a metric space and $A \subset X$, then which of the following conditions imply that A is closed in X: [*a*] A is compact, [*b*] A is complete, [*c*] A is connected.

(*ii*) If A is a closed connected subset of $\mathbf{R}$, is A arcwise connected? Explain why or describe a counterexample.

SOLUTION.

(*i*) The first and the second imply A is closed, but the third does not.■

(*ii*) Yes, for if A is connected and $x, y \in A$ with $x < y$, then the entire closed interval $[x, y]$ is contained in A. One can then join x to y by the linear curve $\gamma(t) = x + t(y - x)$.■

Note. If we replace $\mathbf{R}$ by $\mathbf{R}^n$ for some $n > 1$, the answer is completely different.

2. [*25 points*] Let X be a Hausdorff space, and let $K_1 \supset K_2 \supset \cdots$ be a family of nonempty compact subsets of X. Prove that the intersection $\cap_n K_n$ is also a nonempty compact subset. [*Hint:* The conclusion does not necessarily hold if X is not Hausdorff.]

SOLUTION.

Since X is Hausdorff every subset K_n is closed in X. The family $\{K_n\}$ is thus a family of closed subsets of the compact space K_1, and it has the Finite Intersection Property because $k_{n_1} \cap \cdots \cap K_{n_r}$ is equal to K_m where m is the maximum of the n_i. Thus by compactness we know that the intersection is nonempty. Since the intersection of closed sets is closed, this intersection is also closed in both K_1 and X. Finally, since a closed subspace of a compact space is compact, the intersection is also compact.■

Note. Here is an example to show that the conclusion fails if X is not Hausdorff. Let X be a set with the finite complement topology (the nonempty open subsets are complements of finite sets); then X is compact, and since the subspace topology on an arbitrary $A \subset X$ is also the correspnding finite complement topology, it follows that every subspace is compact. Now take X to be the nonnegative integers $\mathbf{N}$, and let K_n be the complement of $\{0, \cdots, n\}$. Then all the hypotheses of the exercise are satisfied except that X is not Hausdorff, and the intersection $\cap_n K_n$ is empty.■

3. [*25 points*] Let X be a topological space, and let $\mathcal{A} = \{A_\alpha\}$ be a family of nonempty connected subsets such that $X = \cup_\alpha\ A_\alpha$ and for each α, β in the indexing set we have a string of subsets in $\mathcal{A}$ of the form $A_\alpha = A_{\alpha(1)}, \cdots, A_{\alpha(n)} = A_\beta$ such that for $i < n$ we have $A_{\alpha(i)} \cap A_{\alpha(i+1)} \neq \emptyset$. Prove that X is connected. [*Hint:* Connected components provide one approach to proving this.]

SOLUTION.

Let $x \in X$, let C be the connected component containing x, and choose α such that $x \in A_\alpha$. Suppose that $y \in X$, choose β such that $y \in X_\beta$, and also choose a string of subsets $A_\alpha = A_{\alpha(1)}, \cdots, A_{\alpha(n)} = A_\beta$ as in the assumptions for the problem. Since C is the unique maximal connected subset containing x and A_α is a connected subset containing x, we must have $A_\alpha \subset X$. We claim by induction that $A_{\alpha(k)} \subset X$ for each $k \leq n$. The case $k = 1$ follows because $A_{\alpha(1)} = A_\alpha \subset X$. Suppose we know the result for k; to prove it for $k+1$, we use $A_{\alpha(k)} \cap A_{\alpha(k+1)} \neq \emptyset$ to find some z in the intersection. By induction we know that $z \in C$; since $A_{\alpha(k+1)}$ is a connected set containing z and C is the unique maximal such set, it follows that $A_{\alpha(k+1)}$ must be a subset of C. This proves the inductive step and hence that $A_\beta = A_{\alpha(n)} \subset C$. The last sentence implies $y \in C$. Since y was arbitrary, this means that $X \subset C$; but $C \subset X$ by construction and thus $C = X$, which means that X is connected.■

4. [*30 points*] Recall that an isometry of metric spaces is a bijection that preserves distances. If X is a complete metric space and A is a dense subset, then the theorems on completions have the following consequence: *If $h : A \to A$ is an isometry, then there is an isometry $H : X \to X$ such that $H(a) = h(a)$ for all $a \in A$.*

(i) Explain why the mapping H in this result is unique.

(ii) Suppose that $f, g : A \to A$ are isometries. Explain why $g \circ f$ is also an isometry.

(iii) Let $F, G, H : X \to X$ be the isometries associated to f, g and $g \circ f$. Prove that $H = G \circ F$.

SOLUTION.

(i) The values of H on the dense subset A are completely specified by the assumptions. Since the set of points where two continuous maps into a Hausdorff space agree is closed, it follows that the values are uniquely given on $\overline{A} = X$.■

(ii) We need to show that the composite of isometries is an isometry. But

$$\mathbf{d}(g \circ f(u), g \circ f(v)) \quad = \quad \mathbf{d}(f(u), f(v))$$

for all u and v because g is an isometry, and the right hand side is equal to $\mathbf{d}(u, v)$ because f is an isometry. This proves the condition for $g \circ f$ to be an isometry.■

(iii) By construction $H(a) = g(f(a)) = G(F(a))$ for all $a \in A$, so the restrictions of H and $G \circ F$ to A are given by the same isometry $g \circ f$. Apply the first part of the problem to conclude that H and $G \circ F$ must be equal.■

5. [*10 points*] Suppose that $f : X \to Y$ is a continuous open surjection. Prove that f is a quotient map (*i.e.*, the topology on Y is the "quotient topology").

SOLUTION.

We need to show that V is open in Y if and only if its inverse image is open in X. Continuity means that if V is open in Y then $f^{-1}[V]$ is open in X. Conversely if $V \subset Y$ and $f^{-1}[V]$ is open in X, then we may use $V = f(f^{-1}[V])$ and the openness of f to conclude that V is open in Y.■