Math 205A
Fall 2005
R. Schultz

UPDATED GENERAL INFORMATION — NOVEMBER 28, 2005

Reference for projective spaces

Specific examples of compactifications of $\mathbf{R}^n$ known as *projective spaces*, obtained by adding points at infinity for each family of parallel lines, were mentioned in the lectures, and a reference for details was described verbally. For the record, this is all contained in the lecture notes for Mathematics 205C

`http://math.ucr.edu/~res/math205C/lectnotes.*`

in Subsection V.1.1 on pages 165–172. Some of the discussion is beyond the scope of this course (particularly everything about smooth structures), but the main discussion should be understandable to a student who is near the end of Mathematics 205A. The statement that one obtains a compact Hausdorff space appears on page 169. Note that the discussion is formulated to produce spaces $\mathbf{kP}^n$ extending $\mathbf{k}^n$ where $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$.

The discussion in the Mathematics 205C lecture notes does not really show that the image of the standard (continuous open) inclusion h of $\mathbf{k}^n$ in $\mathbf{kP}^n$ is dense, but this can be verified very quickly as follows: Given a point J not in the image of h and an open neighborhood W containing J, we need to find a point $h(x) \in W$. — By construction, $\mathbf{kP}^n$ is a quotient space of $\mathbf{k}^{n+1} - \{0\}$ such that the equivalence class of a point is all nonzero scalar multiples of that point and the image of h is the set of all equivalence classes whose representatives have a nonzero last coordinate. Therefore, if J is not in the image of h then the last coordinate for any vector v representing J must be zero. Suppose that we take an open neighborhood U of J, and we consider its (open) inverse image U^* in $\mathbf{k}^{n+1} - \{0\}$. Then U^* clearly contains a vector w whose last coordinate is nonzero, and if Y is the equivalence class of w, then Y lies in the image of h and also $Y \in U$. Therefore the set $h(\mathbf{k}^n)$ is dense in $\mathbf{kP}^n$.■

Suggestions for the final examination

Here are some suggestions for things to study in connection with the final examination. As usual, this list is furnished with disclaimers that not everything on it will appear on the examination and there might be items on the examination that do not appear to fit into any of these categories (however, at the time of writing the exam has been completed and this list has been compiled with knowledge of the examination's contents).

(1) Know how to prove basic conceptual facts about homeomorphisms like the following: [*a*] If $f : X \to Y$ is a homeomorphism and $A \subset X$, then A is homeomorphic to $f[A]$ (with the subspace topologies in all cases). [*b*] If $f : X \to Y$ is a homeomorphism and X satisfies some standard topological condition (*e.g.*, $\mathbf{T}_i$ for some i, first or second countable, locally compact or connected), then the same is true of Y, and conversely.

(2) Know the four main separation axioms $\mathbf{T}_i$. how they relate to sets of limit points, why they are true for metrizable spaces, whether the class of $\mathbf{T}_i$ spaces is closed under taking

products of subsets (with proofs except when $i = 4$). Be able to provide brief (≤ 4 line) descriptions of spaces that are not $\mathbf{T}_1$, $\mathbf{T}_1$ but not $\mathbf{T}_2$, and $\mathbf{T}_2$ but not metrizable (for example, in the latter case a space that is separable but not second countable).

(3) Know the implications of metrizability for functions on compact and connected metric spaces, and also their implications for the space (*e.g.*, the Lebesgue covering lemma, and more simply for a fixed point in a compact metric space there is an upper bound on the distances of other points from the given one).

(4) Understand the definition and basic properties of dense, nowhere dense and meager subsets, the statement of Baire's Theorem, and know examples of metric spaces whose topologies do not support complete metrics. In the case of dense sets, know equivalent characterizations (just the statements).

(5) Know the logical relationships between connectedness and its variants (local connectedness and arcwise connectedness) and specifically how these concepts are related for basic types of subsets of Euclidean spaces (*e.g.*, open and closed subsets).

(6) Understand how basic properties like compactness, second countability, separability, the Lindelöf property, nowhere denseness and meagerness behave under taking finite or countably infinite unions of subspaces.

(7) Understand the basic results on connected components of a topological space as well as the corresponding notion of arc component and the logical relationships between the two concepts. Also know how to find examples of different topologies on the same set which yield the same infinite set of connected components, but the latter are open in one topology but not the other (the simplest case would be when the connected components are all singletons).

(8) Know the definition of quotient topology, and given a continuous surjection $f : X \to Y$, know how to determine whether a given topology is the quotient topology by means of the definitions, know how to use continuous mappings to recognize that one compact metric space is homeomorphic to a quotient of another, and know how to construct examples where the original space is metrizable but the quotient is not even $\mathbf{T}_1$,

(9) Know the equivalent definitions for a locally compact $\mathbf{T}_2$ space, conditions for a subset of such a space to have this same property, the construction of the one point compactification. Know how to prove the recognition principle for the latter and how it applies to describing the one point compactification of $\mathbf{R}^n$.

SPECIFIC EXERCISES. Here are some additional problems that might be worth studying:

1. Suppose that X is a metric space that is uniformly locally compact in the sense that there is some $\delta > 0$ such that for each $x \in X$ the neighborhood $N_\delta(x)$ has compact closure. Prove that X is complete. Explain why the conclusion fails if one only assumes that for each x there is some $\delta(x) > 0$ with the given property (give an example).

2. Suppose that X is a noncompact locally compact $\mathbf{T}_2$ space and A is a noncompact closed subset of X. Prove that the one point compactification $A^\bullet$ is homeomorphic to a subset of the one point compactification $X^\bullet$.

3. Is it possible to find compact metric spaces X and Y such that neither consists of a single point, neither is homeomorphic to the other, but each is homeomorphic to a quotient of the other? [*Hint:* See point number 8 in the preceding list, and consider the main example of Section 44 in Munkres.]

4. Given a topological space S and a nonempty closed subspace T, we define S/T, verbalized as S modulo T, to be the quotient space whose equivalence classes consist of all $\{x\}$ such that $x \in S - T$ and the set T itself. — Suppose that we are given a continuous map of topological spaces $f : X \to Y$, and that $A \subset X$ and $B \subset Y$ are nonempty closed subsets satisfying $f[A] \subset B$. Prove that there is a unique continuous map $F : X/A \to Y/B$ such that for all $\mathbf{c} \in X/A$, if $\mathbf{c}$ is the equivalence class of $x \in X$, then $F(\mathbf{c})$ is the equivalence class of $f(x)$.

As noted in previous verbal announcements, the final examination is intended to be double the length of a midterm examination with 8 problems worth a total of 200 points, but there will be the standard three hour time period to work on it. Three of the problems will cover material from the previous examinations, three will cover new material, and the remaining two will be a mixture of both of the preceding types.