Standard forms for writing numbers

In order to relate the abstract mathematical descriptions of familiar number systems to the everyday descriptions of numbers by decimal expansions and similar means, it is necessary to verify that objects the mathematically constructed number systems can be written in the standard forms and that these forms have the expected properties. The purpose of this document is to carry out the details of such verifications.

Not surprisingly, our discussion will be based upon the standard axiomatic description of the real number system, which is discussed in the file(s)

http://math.ucr.edu/~res/math205A/realnumbers.doc
http://math.ucr.edu/~res/math205A/realnumbers.pdf
http://math.ucr.edu/~res/math205A/realnumbers.ps

http://math.ucr.edu/~res/math205A/uniqreals.doc
http://math.ucr.edu/~res/math205A/uniqreals.pdf
http://math.ucr.edu/~res/math205A/uniqreals.ps

in the course directory. The basic properties may be summarized as follows:

1. The real numbers form an ***ordered field***, and it is ***complete*** with respect to the ordering.

2. The subset of ***natural numbers***, which is the smallest subset that contains 0 and 1 and is closed under addition, is ***well – ordered***; *i.e.*, every nonempty subset contains a (unique) least element.

We shall summarize some consequences of these properties that are important for our purposes.

Archimedean Law. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are positive real numbers, then there is a positive integer* $\mathbf{n}$ *such that* $\mathbf{n\,a > b}$.

By the well – ordering of the positive integers, there is a *(unique) minimal value* of $\mathbf{n}$ for which this holds.

Infinite series convergence criterion. *If* $\{\mathbf{a_n}\}$ *is a sequence of nonnegative terms such that the partial sums* $\mathbf{s_n = a_1 + \dots + a_n}$ *is bounded by some constant* $\mathbf{K}$ *that does not depend on* $\mathbf{n}$, *then the infinite series* $\mathbf{a_1 + a_2 + \dots + a_n + \dots}$ *converges (to a finite nonnegative value)*.

In fact, the sum of the infinite series is the least upper bound of the partial sums $\mathbf{s_n}$. The most fundamental example of this sort is the classical *geometric series* with terms $\mathbf{a_n = r^{n-1}}$, and in this case we know the sum is equal to $1 / (1 - \mathbf{r})$.

One of the most elementary facts about a positive real number $\mathbf{x}$ is that it can be written as the sum $\mathbf{[x] + (x)}$ of a nonnegative integer $\mathbf{[x]}$ and a nonnegative real number $\mathbf{(x)}$

that is strictly less than one, and this decomposition is unique. The integer **[x]** is often called the ***greatest integer function*** of **x** or the ***characteristic*** of **x**, and the remaining number **(x)** is often called the ***fractional part*** or ***mantissa*** of **x**. The characteristic – mantissa terminology dates back to the original tables of base 10 logarithms published by H. Briggs (1561 – 1630); the literal meaning of the Latin root word *mantisa* is "makeweight," and it denotes something small that is placed onto a scale to bring the weight up to a desired value. We shall derive the decomposition of a nonnegative real number into a characteristic and mantissa from the axiomatic properties of the real numbers.

Theorem. *Let* **r** *be an arbitrary nonnegative real number. Then there is a unique decomposition of* **r** *as a sum* $\mathbf{n} + \mathbf{s}$ *where* **n** *is a nonnegative integer and* $0 \le \mathbf{s} < 1$.

Proof. By the Archimedean Law there is a nonnegative integer **m** such that $\mathbf{m} > \mathbf{r}$, and since the nonnegative integers are well – ordered there is a minimum such integer . Since **r** is nonnegative it follows that $\mathbf{m_1}$ cannot be zero and hence must also be positive. Therefore $\mathbf{m_1} - 1$ is also nonnegative and by the minimality of the positive integer $\mathbf{m_1}$ we must have $\mathbf{m_1} - 1 \le \mathbf{r}$. . If we take $\mathbf{n} = \mathbf{m_1} - \mathbf{1}$ and $\mathbf{s} = \mathbf{r} - \mathbf{n}$ then we have $\mathbf{r} = \mathbf{n} + \mathbf{s}$ where **n** and **s** have the desired properties. Suppose that we also have $\mathbf{r} = \mathbf{q} + \mathbf{v}$ where **q** is a nonnegative integer and $0 \le \mathbf{q} < 1$. By hypothesis we have

$$\mathbf{q} \le \mathbf{r} < \mathbf{q} + 1$$

and the right hand inequality implies $\mathbf{n} + 1 \le \mathbf{q} + 1$, or equivalently $\mathbf{n} \le \mathbf{q}$. The equation $\mathbf{r} = \mathbf{n} + \mathbf{s} = \mathbf{q} + \mathbf{v}$ can therefore be rewritten in the form

$$0 \le \mathbf{q} - \mathbf{n} = \mathbf{s} - \mathbf{v}$$

and since (*i*) $\mathbf{s} - \mathbf{v} \le \mathbf{s} < \mathbf{1}$ and (*ii*) $\mathbf{q} - \mathbf{n}$ is an integer, it follows that $\mathbf{n} = \mathbf{q}$ and $\mathbf{s} = \mathbf{v}$.

Base N expansions for natural numbers

We shall make extensive use of the **Long Division Property** of natural numbers:

Division Theorem. *Given two natural numbers* **a** *and* **b** *with* $\mathbf{b} > 1$, *there are* ***unique*** *natural numbers* **q** *and* **r** *such that* $\mathbf{a} = \mathbf{b}\,\mathbf{q} + \mathbf{r}$, *where* $0 \le \mathbf{r} \le \mathbf{b} - 1$.

The numbers **q** and **r** are often called the *integral quotient* and *remainder* respectively.

Here is the standard result on ***base*** **N** or **N –** ***adic*** expansions of positive integers. In the standard case when $\mathbf{N} = 10$, this yields the standard way of writing a nonnegative integer in terms of the usual Hindu – Arabic numerals, while if $\mathbf{n} = 2$ or 8 or 16 this yields the binary or octal or hexadecimal expansion respectively.

Theorem. *Let* **k** *be a positive integer, and let* $\mathbf{N} > 1$ *be another positive integer. Then there are unique integers* $\mathbf{a_j}$ *such that* $0 \le \mathbf{a_j} \le \mathbf{N} - 1$ *and*

$$\mathbf{k} = \mathbf{a_0} + \mathbf{a_1}\,\mathbf{N} + \ldots + \mathbf{a_m}\,\mathbf{N^m}$$

for a suitable nonnegative integer $\mathbf{m}$*.*

In the course of proving this result it will be useful to know the following:

Lemma. *Suppose that integers* $\mathbf{N}$, $\mathbf{k}$, *and* $\mathbf{a}_j$ *are given as above. Then we have*

$$\mathbf{a}_0 + \mathbf{a}_1 \mathbf{N} + \ldots + \mathbf{a}_m \mathbf{N}^m \leq \mathbf{N}^{m+1}.$$

Proof of Lemma. Since $\mathbf{a}_j \leq \mathbf{N} - 1$ for each $\mathbf{j}$ we have

$$\mathbf{a}_j \mathbf{N}^j \leq (\mathbf{N} - 1)\mathbf{N}^j = \mathbf{N}^{j+1} - \mathbf{N}^j$$

and therefore we have the inequality

$$\mathbf{a}_0 + \mathbf{a}_1 \mathbf{N} + \ldots + \mathbf{a}_m \mathbf{N}^m \leq \mathbf{N} - 1 + (\mathbf{N}^2 - \mathbf{N}) + \ldots + (\mathbf{N}^{m+1} - \mathbf{N}^m) =$$

$$\mathbf{N}^{m+1} - 1 < \mathbf{N}^{m+1}.$$

Proof of Theorem. It is always possible to find an exponent $\mathbf{q}$ such that $2^q > \mathbf{k}$, and since $\mathbf{k} \geq 2$ it follows that we also have $\mathbf{N}^q > 2^q > \mathbf{k}$. Let $[\, S_m \,]$ be the statement of the statement that every positive integer less than $\mathbf{N}^{m+1}$ has a unique expression as above. If $\mathbf{m} = 0$ then the result follows immediately from the long division theorem, for then $\mathbf{k} = \mathbf{a}_0$. Suppose now that $[\, S_{p-1} \,]$ is true and consider the statement $[\, S_p \,]$. If $\mathbf{k} < \mathbf{N}^{p+1}$ then we can use long division to write $\mathbf{k}$ uniquely in the form

$$\mathbf{k} = \mathbf{k}_0 + \mathbf{a}_p \mathbf{N}^p$$

where $\mathbf{a}_p \geq 0$ and $0 \leq \mathbf{k}_0 < \mathbf{N}^p$. We claim that $\mathbf{a}_p < \mathbf{N}$. If this were false then we would have $\mathbf{k} \geq \mathbf{a}_p \mathbf{N}^p \geq \mathbf{N}\,\mathbf{N}^p = \mathbf{N}^{p+1}$ and this contradicts the assumption that $\mathbf{k} < \mathbf{N}^{p+1}$.

By induction we know that $\mathbf{k}_0$ has a unique expression as a sum

$$\mathbf{k}_0 = \mathbf{a}_0 + \mathbf{a}_1 \mathbf{N} + \ldots + \mathbf{a}_{p-1} \mathbf{N}^{p-1}$$

for suitable $\mathbf{a}_j$. This proves existence. To prove uniqueness, suppose that we have

$$\mathbf{k} = \mathbf{a}_0 + \mathbf{a}_1 \mathbf{N} + \ldots + \mathbf{a}_p \mathbf{N}^p = \mathbf{b}_0 + \mathbf{b}_1 \mathbf{N} + \ldots + \mathbf{b}_p \mathbf{N}^p.$$

Denote all but the last terms of these sums by $\mathbf{A} = \mathbf{a}_0 + \mathbf{a}_1 \mathbf{N} + \ldots + \mathbf{a}_{p-1} \mathbf{N}^{p-1}$ and $\mathbf{B} = \mathbf{b}_0 + \mathbf{b}_1 \mathbf{N} + \ldots + \mathbf{b}_{p-1} \mathbf{N}^{p-1}$. Then $0 \leq \mathbf{A}, \mathbf{B} \leq \mathbf{N}^p - 1$ by the lemma, and therefore by the uniqueness of the long division expansion of k it follows that $\mathbf{a}_p = \mathbf{b}_p$ and $\mathbf{A} = \mathbf{B}$. By the induction hypothesis the latter implies that $\mathbf{a}_j = \mathbf{b}_j$ for all $\mathbf{j} < \mathbf{p}$. Therefore we have also shown uniqueness.

Decimal expansions for real numbers

As noted in the online document(s) **realnumbers. ***, a mathematically sound definition of the real numbers should yield the usual decimal expansions for base 10 as well as the corresponding expansions for other choices of the base **N**. We shall verify this here and show that decimal expansions have several properties that are well – known from our everyday experience in working with decimals.

Although decimal expansions of real numbers are extremely useful for computational purposes, they are not particularly convenient for theoretical or conceptual purposes. For example, although every nonzero real number should have a reciprocal, describing this reciprocal completely and explicitly by infinite decimal expansions is awkward and generally unrealistic. Another difficulty is that decimal expansions are not necessarily unique; for example, the relation

$$1.0 = 0.9999999\ldots$$

reflects the classical geometric series formula

$$\mathbf{a}/(1-\mathbf{r}) = \mathbf{a} + \mathbf{a}\,\mathbf{r} + \mathbf{a}\,\mathbf{r}^2 + \ldots + \mathbf{a}\,\mathbf{r}^{\mathbf{k}} + \ldots$$

when $\mathbf{a} = 9/10$ and $\mathbf{r} = 1/10$. A third issue, which is mentioned in the document(s) **realnumbers.***, is whether one gets an equivalent number system if one switches from base 10 arithmetic to some other base. It is natural to expect that the answer to this question is yes, but any attempt to establish this directly runs into all sorts of difficulties almost immediately. This is not purely a theoretical problem; the use of digital computers to carry out numerical computations implicitly assumes that one can work with real numbers equally well using infinite expansions with base 2 (or base 8 or 16 as in many computer codes, or even base 60 as in ancient Babylonian mathematics). One test of the usefulness of the abstract approach to real numbers is whether it yields such consequences.

A natural starting point is to verify that infinite decimal expansions always yield real numbers.

Decimal Expansion Theorem. *Every infinite series of real numbers having the form*

$$\mathbf{a}_{\mathbf{N}}10^{\mathbf{N}} + \mathbf{a}_{\mathbf{N}-1}10^{\mathbf{N}-1} + \ldots + \mathbf{a}_0 + \mathbf{b}_1 10^{-1} + \mathbf{b}_2 10^{-2} + \ldots + \mathbf{b}_{\mathbf{k}} 10^{-\mathbf{k}} + \ldots$$

$$(\textit{with } 0 \le \mathbf{a}_{\mathbf{i}}, \mathbf{b}_{\mathbf{j}} \le 9)$$

is convergent. *Conversely, every positive real number is the sum of an infinite series of this type where the coefficients of the powers of* 10 *are integers satisfying the basic inequalities* $0 \le \mathbf{a}_{\mathbf{i}}, \mathbf{b}_{\mathbf{j}} \le 9$.

As noted above, there are two ways of writing 1 as an infinite series of this type, so such a representation is not unique, but there is only one way of expressing a number as such a sum for which infinitely many coefficients are nonzero. Of course, we eventually want to prove this generates all ambiguities in decimal expansions; this will be done later.

PROOF OF THE DECIMAL EXPANSION THEOREM. The proof of this result splits naturally into two parts, one for each implication direction.

Formal infinite decimal expansions determine real numbers: If one can show this for positive decimal expansions, it will follow easily for negative ones as well, so we shall restrict attention to the positive case. Consider the formal expression given above:

$$(\mathbf{a_N}10^{\mathbf{N}} + \mathbf{a_{N-1}}10^{\mathbf{N-1}} + \ldots + \mathbf{a}_0 + \mathbf{b}_1 10^{-1} + \mathbf{b}_2 10^{-2} + \ldots + \mathbf{b_k}10^{-\mathbf{k}} + \ldots)$$

For each integer $\mathbf{p} > 0$, define $\mathbf{s_p}$ to be the sum of all terms in this expression up to and including $\mathbf{b_p}10^{-\mathbf{p}}$ and let $\mathbf{S}$ be the set of all such numbers $\mathbf{s_p}$. Then the set $\mathbf{S}$ has an upper bound, and in fact we claim that $10^{\mathbf{N}+1}$ is an upper bound for $\mathbf{S}$. To see this, observe that $\mathbf{a_N}\,10^{\mathbf{N}} + \mathbf{a_{N-1}}10^{\mathbf{N-1}} + \ldots + \mathbf{a}_0 \le 10^{\mathbf{N}+1} - 1$ by a previous lemma and

$$\mathbf{b}_1 10^{-1} + \mathbf{b}_2 10^{-2} + \ldots + \mathbf{b_k}10^{-\mathbf{k}} + \ldots \le 9\,(10^{-1} + 10^{-2} + \ldots + 10^{-\mathbf{k}} + \ldots) = 1$$

and the assertion about an upper bound follows immediately from this. The least upper bound $\mathbf{r}$ for $\mathbf{S}$ turns out to be the limit of the sequence of partial sums $\{\mathbf{s_p}\}$.

Real numbers determine infinite decimal expansions: Given (say) a positive real number $\boldsymbol{r}$, the basic idea is to find a sequence of finite decimal fractions $\{\mathbf{s_p}\}$ such that for every value of $\mathbf{p}$ the number $\mathbf{s_p}$ is expressible as a fraction whose denominator is given by $10^{\mathbf{p}}$ and

$$\mathbf{s_p} \le \mathbf{r} < \mathbf{s_p} + 10^{-\mathbf{p}}.$$

More precisely, suppose that we already have $\mathbf{s_p}$ and we want to find the next term. By construction $10^{\mathbf{p}}\mathbf{s_p}$ is a positive integer and $10^{\mathbf{p}}\mathbf{s_p} \le 10^{\mathbf{p}}\mathbf{r} < 10^{\mathbf{p}}\mathbf{s_p} + 1$, so that

$$10^{\mathbf{p}+1}\mathbf{s_p} \le 10^{\mathbf{p}+1}\mathbf{r} < 10^{\mathbf{p}+1}\mathbf{s_p} + 10.$$

Choose $\mathbf{b_{p+1}}$ to be the largest integer such that

$$\mathbf{b_{p+1}} \le 10^{\mathbf{p}+1}\mathbf{r} - 10^{\mathbf{p}+1}\mathbf{s_p}.$$

The right hand side is positive so this means that $\mathbf{b_{p+1}} \ge 0$. On the other hand, the previous inequalities also show that $\mathbf{b_{p+1}} < 10$ and since $\mathbf{b_{p+1}}$ is an integer this implies $\mathbf{b_{p+1}} \le 9$. If we now take $\mathbf{s_{p+1}} = 10\,\mathbf{s_p} + \mathbf{b_{p+1}}$ then it will follow that

$$\mathbf{s_{p+1}} \le \mathbf{r} < \mathbf{s_{p+1}} + 10^{-(\mathbf{p}+1)}.$$

To see that the sequence converges, note that it corresponds to the infinite series

$$\mathbf{s_p} + \Sigma_{\mathbf{p}}\,(\mathbf{b_{p+1}}\,10^{-\mathbf{p}}),$$

which converges by a comparison with the modified geometric series $\mathbf{s_p} + \Sigma_{\mathbf{p}}\,10^{(1-\mathbf{p})}$.

Scientific notation. The standard results on writing positive real numbers in ***scientific notation*** follow immediately from the preceding considerations.

Theorem. *Every positive real number has a unique expansion of the form* $\mathbf{a} \cdot 10^{\mathbf{M}}$*, where* $1 \leq \mathbf{a} < 10$ *and* **M** *is an integer.*

As is always the case for unique existence results, the proof splits into two parts.

Existence. If **x** has the decimal expansion

$$\mathbf{a}_{\mathbf{N}} \cdot 10^{\mathbf{N}} + \mathbf{a}_{\mathbf{N-1}} \cdot 10^{\mathbf{N-1}} + \ldots + \mathbf{a}_0 + \mathbf{b}_1 \cdot 10^{-1} + \mathbf{b}_2 \cdot 10^{-2} + \ldots + \mathbf{b}_k \cdot 10^{-\mathbf{k}} + \ldots$$

$$(\textit{with } 0 \leq \mathbf{a}_i, \mathbf{b}_j \leq 9)$$

then $\mathbf{x} \cdot 10^{-\mathbf{N}}$ lies in the interval [1, 10) by construction.

Uniqueness. Suppose that we can write **x** as $\mathbf{a} \cdot 10^{\mathbf{M}}$ and $\mathbf{b} \cdot 10^{\mathbf{N}}$. Then by the conditions on the coefficients, we know that $\mathbf{x} \in [10^{\mathbf{M}}, 10^{\mathbf{M+1}}) \cap [10^{\mathbf{N}}, 10^{\mathbf{N+1}})$. Since the half open intervals $[10^{\mathbf{M}}, 10^{\mathbf{M+1}})$ and $[10^{\mathbf{N}}, 10^{\mathbf{N+1}})$ are disjoint unless $\mathbf{M} = \mathbf{N}$, it follows that the latter must hold. Therefore the equations $\mathbf{x} = \mathbf{a} \cdot 10^{\mathbf{M}} = \mathbf{b} \cdot 10^{\mathbf{N}}$ and $\mathbf{M} = \mathbf{N}$ imply $\mathbf{a} = \mathbf{b}$.

Decimal expansions of rational numbers

In working with decimals one eventually notices that the decimal expansions for rational numbers have the following special property:

Eventual Periodicity Property. *Let* **r** *be a rational number such that* $0 < \mathbf{r} < 1$*, and let*

$$\mathbf{r} = \mathbf{b}_1 10^{-1} + \mathbf{b}_2 10^{-2} + \ldots + \mathbf{b}_k 10^{-\mathbf{k}} + \ldots$$

be a decimal expansion. Then the sequence $\{ \mathbf{b}_k \}$ *is* ***eventually periodic*** ; *i.e., there are positive integers* **M** *and* **Q** *such that* $\mathbf{b}_k = \mathbf{b}_{k+Q}$ *for all* $\mathbf{k} > \mathbf{M}$.

Proof. Let $\mathbf{a} / \mathbf{b}$ be a rational number between 0 and 1, where **a** and **b** are integers satisfying $0 < \mathbf{a} < \mathbf{b}$. Define sequences of numbers $\mathbf{r}_n$ and $\mathbf{x}_n$ recursively, beginning with $\mathbf{r}_0 = \mathbf{a}$ and $\mathbf{x}_0 = 0$. Given $\mathbf{r}_n$ and $\mathbf{x}_n$ express the product $10\,\mathbf{r}_n$ by long division in the form $10\,\mathbf{r}_n = \mathbf{b}\,\mathbf{x}_{n+1} + \mathbf{r}_{n+1}$ where $\mathbf{x}_{n+1} \geq 0$ and $0 \leq \mathbf{r}_{n+1} < \mathbf{b}$.

CLAIMS:

1. *Both of these numbers only depend upon* $\mathbf{r}_n$.
2. *We have* $\mathbf{x}_{n+1} < 10$.

The first part is immediate from the definition in terms of long division, and to see the second note that $\mathbf{x}_{n+1} \geq 10$ would imply $10\,\mathbf{r}_n \geq 10\,\mathbf{b}$, which contradicts the fundamental remainder condition $\mathbf{r}_n < \mathbf{b}$.

Since $\mathbf{r_n}$ can only take integral values between 0 and $\mathbf{b} - 1$, it follows that there are some numbers $\mathbf{Q}$ and m such that $\mathbf{r_m} = \mathbf{r_{m+Q}}$.

CLAIM: $\mathbf{r_k} = \mathbf{r_{k+Q}}$ *for all* $\mathbf{k} \geq \mathbf{m}$.

We already know this for $\mathbf{p} = \mathbf{m}$, so assume it is true for $\mathbf{p} \leq \mathbf{k}$. Now each term in the sequence $\mathbf{r_n}$ depends only on the previous term, and hence the relation $\mathbf{r_k} = \mathbf{r_{k+Q}}$ implies $\mathbf{r_{k+1}} = \mathbf{r_{k+Q+1}}$. Therefore the claim is true by finite induction.

CONVERSELY, suppose that the statement in the claim holds for the decimal expansion of some number, and choose **m** and **Q** as above. Let **s** be given by the first $\mathbf{m} - 1$ terms in the decimal expansion of **y**, and let **t** be the sum of the next **Q** terms. It then follows that **y** is equal to $\mathbf{s} + \mathbf{t}\,(1 + 10^{-Q} + 10^{-2Q} + 10^{-3Q} + \ldots)$. Now **s, t** and the geometric series in parentheses are all rational numbers, and therefore it follows that **y** is also a rational number. Therefore we have the following result:

Theorem. *A real number between* 0 *and* 1 *has a decimal expansion that is eventually periodic if and only if it is a rational number.*

It is easy to find examples illustrating the theorem:

$$1/3 = 0.33333333333333333333333333333333\ldots$$

$$1/6 = 0.16666666666666666666666666666666\ldots$$

$$1/7 = 0.14285714285714285714285714285714285 7\ldots$$

$$1/11 = 0.01010101010101010101010101010101\ldots$$

$$1/12 = 0.08333333333333333333333333333333\ldots$$

$$1/13 = 0.07692307692307692307692307692307692 3\ldots$$

$$1/17 = 0.05882352941176470588235294117647058 8\ldots$$

$$1/18 = 0.05555555555555555555555555555555\ldots$$

$$1/19 = 0.05263157894736842105263157894736842 1\ldots$$

$$1/23 = 0.04347826086956521739130434782608769 5\ldots$$

$$1/27 = 0.03703703703703703703703703703703703 7\ldots$$

$$1/29 = 0.03448275862068965517241379310344827 5\ldots$$

$$1/31 = 0.03225806451612903225806451612903225 8\ldots$$

$$1/34 = 0.02941176470588235294117647058823529 4\ldots$$

$$1/37 = 0.02702702702702702702702702702702702 7\ldots$$

Note that the minimal period lengths in these examples are 1, 1, 6, 2, 1, 6, 16, 1, 18, 22, 3, 28, 15, 16 and 3. One is naturally led to the following question:

Given a fraction **a** / **b** *between* 0 *and* 1**,** *what determines the (minimal) period length* **Q**?

To illustrate the ideas, we shall restrict attention to the special case where **a** / **b** **=** 1 / **p,** where **p** is a prime not equal to 2 or 5 (the two prime divisors of 10). In this case the methods of abstract algebra yield the following result:

Theorem. *If* **p** $\neq$ 2**,** 5 *is a prime, then the least period* **Q** *is for the decimal expansion of* 1 / **p** *is equal to the multiplicative order of* 10 *in the (finite cyclic) group of multiplicative units for the integers mod* **p.**

We shall not verify this result here, but the proof is not difficult.

Corollary. *The least period* **Q** *divides* **p –** 1**.**

The corollary follows because the order of the group of units is equal to **p –** 1 and the order of an element in a finite group always divides the order of the group.

One is now led to ask when the period is actually equal to this maximum possible value. Our examples show this is true for the primes 7, 19, 23 and 29 but not for the primes 11, 13, 31 or 37**.**

More generally, one can define a primitive root of unity in the integers mod **p** to be an integer **a** mod **p** such that **a** is not divisible by **p** and the multiplicative order of the class of **a** in the integers mod p is precisely **p –** 1**.** Since the group of units is cyclic, such primitive roots always exist, and one can use the concept of primitive root to rephrase the question about maximum periods for decimal expansions in the following terms:

For which primes **p** *is* 10 *a primitive root of unity mod* **p?**

A simple answer to this question does not seem to exist. In the nineteen twenties E. Artin (1898 – 1962) stated the following conjecture:

Every integer **a** **>** 1 *is a primitive root of unity mod* **p** *for infinitely many primes* **p.**

This means that 10 ***should*** be the primitive root for infinitely many primes **p**, and hence there should be infinitely many full – period primes. Quantitatively, the conjecture amounts to showing that about 37% of all primes asymptotically have 10 as primitive root. The percentage is really an approximation to Artin's constant

$$C_{\text{Artin}} = \prod_{k=1}^{\infty} \left[1 - \frac{1}{p_k(p_k - 1)}\right] = 0.3739558136\ldots$$

where $\mathbf{p_k}$ denotes the $\mathbf{k^{th}}$ prime. Further information about this number and related topics appears in the following online reference:

http://mathworld.wolfram.com/ArtinsConstant.html

Uniqueness of decimal expansions

The criterion for two decimal expressions to be equal is well understood.

Theorem. *Suppose that we are given two decimal expansions that yield the same real number:*

$$\mathbf{a_N}10^{\mathbf{N}} + \mathbf{a}_{\mathbf{N}-1}10^{\mathbf{N}-1} + \ldots + \mathbf{a}_0 + \mathbf{b}_1 10^{-1} + \mathbf{b}_2 10^{-2} + \ldots + \mathbf{b_k}10^{-\mathbf{k}} + \ldots =$$

$$\mathbf{c_N}10^{\mathbf{N}} + \mathbf{c}_{\mathbf{N}-1}10^{\mathbf{N}-1} + \ldots + \mathbf{c}_0 + \mathbf{d}_1 10^{-1} + \mathbf{d}_2 10^{-2} + \ldots + \mathbf{d_k}10^{-\mathbf{k}} + \ldots$$

Then $\mathbf{a}_j = \mathbf{c}_j$ *for all* $\mathbf{j}$*, and one of the following is also true:*

1. *For each* $\mathbf{k}$ *we have* $\mathbf{b_k} = \mathbf{d_k}$.
2. *There is an* $\mathbf{L} > 0$ *such that* $\mathbf{b_k} = \mathbf{d_k}$ *for every* $\mathbf{k} < \mathbf{L}$ *but* $\mathbf{b}_{\mathbf{L}+1} = \mathbf{d}_{\mathbf{L}} + 1$, *while* $\mathbf{b_k} = 0$ *for all* $\mathbf{k} > \mathbf{L}$ *and* $\mathbf{d_k} = 9$ *for all* $\mathbf{k} > \mathbf{L}$.
3. *There is an* $\mathbf{L} > 0$ *such that* $\mathbf{b_k} = \mathbf{d_k}$ *for every* $\mathbf{k} < \mathbf{L}$ *but* $\mathbf{d}_{\mathbf{L}+1} = \mathbf{b}_{\mathbf{L}} + 1$, *while* $\mathbf{d_k} = 0$ *for all* $\mathbf{k} > \mathbf{L}$ *and* $\mathbf{b_k} = 9$ *for all* $\mathbf{k} > \mathbf{L}$ *(the opposite of the previous possibility).*

If $\mathbf{x}$ and $\mathbf{y}$ are given by the respective decimal expansions above, then $\mathbf{x} = \mathbf{y}$ implies the greatest integer functions satisfy $[\mathbf{x}] = [\mathbf{y}]$, which in turn implies that $\mathbf{a}_j = \mathbf{c}_{j\,j}$ for all $\mathbf{j}$. Furthermore, we then also have $(\mathbf{x}) = (\mathbf{y})$ and accordingly the proof reduces to showing the result for numbers that are between 0 and 1.

The following special uniqueness result will be helpful at one point in the general proof.

Lemma. *For each positive integer* $\mathbf{k}$ *let* $\mathbf{t}_k$ *be an integer between* 0 *and* 9. *Then we have*

$$1 = \mathbf{t}_1 10^{-1} + \mathbf{t}_2 10^{-2} + \ldots + \mathbf{t_k}10^{-\mathbf{k}} + \ldots$$

if and only if $\mathbf{t}_k = \mathbf{9}$ *for all* $\mathbf{k}$.

Proof. Let $\mathbf{t}$ be the summation on the right hand side. If $\mathbf{t}_k = \mathbf{9}$ for all $\mathbf{k}$ then $\mathbf{t} = 1$ by the geometric series formula. Conversely, if $\mathbf{t}_m < \mathbf{9}$ for a specific value of $\mathbf{m}$ then

$$\mathbf{t}_1 10^{-1} + \mathbf{t}_2 10^{-2} + \ldots + \mathbf{t_k}10^{-\mathbf{k}} + \ldots < \mathbf{u}_1 10^{-1} + \mathbf{u}_2 10^{-2} + \ldots + \mathbf{u_k}10^{-\mathbf{k}} + \ldots$$

where $\mathbf{u_k} = 9$ for $\mathbf{k} \neq \mathbf{m}$ and $\mathbf{u}_m \leq 8$. The latter implies that the right hand side is less than or equal to $1 - 10^{-\mathbf{m}}$, which is strictly less than 1.

Theorem. *If we are given two decimal expansions*

$$\mathbf{x} = \mathbf{x}_1 10^{-1} + \mathbf{x}_2 10^{-2} + \ldots + \mathbf{x_k} 10^{-\mathbf{k}} + \ldots$$

$$\mathbf{y} = \mathbf{y}_1 10^{-1} + \mathbf{y}_2 10^{-2} + \ldots + \mathbf{y_k} 10^{-\mathbf{k}} + \ldots$$

then $\mathbf{x} = \mathbf{y}$ *if and only if one of the following is true*:

1. *For all positive integers* $\mathbf{k}$ *we have* $\mathbf{x}_k = \mathbf{y}_k$.
2. *There is some positive integer* $\mathbf{M}$ *such that* **[i]** $\mathbf{x}_k = \mathbf{y}_k$ *for all* $\mathbf{k} < \mathbf{M}$, **[ii]** $\mathbf{x}_M = \mathbf{y}_M + 1$, **[iii]** $\mathbf{x}_k = 0$ *for* $\mathbf{k} > \mathbf{M}$, *and* **[iv]** $\mathbf{y}_k = 9$ *for* $\mathbf{k} > \mathbf{M}$.
3. *A corresponding statement holds in which the roles of* $\mathbf{x}_k$ *and* $\mathbf{y}_k$ *are interchanged: There is some positive integer* $\mathbf{M}$ *such that* **[i]** $\mathbf{x}_k = \mathbf{y}_k$ *for all* $\mathbf{k} < \mathbf{M}$, **[ii]** $\mathbf{y}_M = \mathbf{x}_M + 1$, **[iii]** $\mathbf{y}_k = 0$ *for* $\mathbf{k} > \mathbf{M}$, *and* **[iv]** $\mathbf{x}_k = 9$ *for* $\mathbf{k} > \mathbf{M}$.

Proof. Suppose that the first alternative does not happen, and let $\mathbf{L}$ be the first positive integer such that $\mathbf{x}_L \neq \mathbf{y}_L$. Without loss of generality, we may as well assume that the inequality is $\mathbf{x}_L > \mathbf{x}_L$ (if the inequality points in the opposite direction, then one can apply the same argument reversing the roles of $\mathbf{x}_k$ and $\mathbf{x}_k$ throughout). Let z be given by the first $\mathbf{L} - 1$ terms of either $\mathbf{x}$ or $\mathbf{y}$ (these are equal).

CASE 1. Suppose that $\mathbf{x}_L \geq \mathbf{y}_L + 2$. Note that $\mathbf{y}_L \leq 7$ is true in this case. We then have

$$\mathbf{y} \leq \mathbf{z} + 10^{-\mathbf{L}}\mathbf{y}_L + \mathbf{9} \times 10^{-\mathbf{L}}(10^{-1} + 10^{-2} + \ldots + 10^{-\mathbf{k}} + \ldots) = \mathbf{z} + 10^{-\mathbf{L}}(\mathbf{y}_L + \mathbf{1}) <$$

$$\mathbf{z} + 10^{-\mathbf{L}}(\mathbf{x_L}) \leq \mathbf{z} + 10^{-\mathbf{L}}(\mathbf{x_L} + \mathbf{x}_{L+1}10^{-1} + \mathbf{x}_{L+2}10^{-2} + \ldots + \mathbf{x_{L+k}}10^{-\mathbf{k}} + \ldots) = \mathbf{x}.$$

Therefore $\mathbf{x} > \mathbf{y}$ if we have $\mathbf{x}_L \geq \mathbf{y}_L + 2$.

CASE 2. Suppose that $\mathbf{x}_L = \mathbf{y}_L + 1$, and let $\mathbf{w} = 10^{-\mathbf{L}}\mathbf{y}_L$, so that $\mathbf{x}_L = \mathbf{w} + 10^{-\mathbf{L}}$. We may then write

$$\mathbf{x} = \mathbf{z} + (\mathbf{w} + 10^{-\mathbf{L}}) + 10^{-\mathbf{L}}\mathbf{u} \quad \text{and} \quad \mathbf{y} = \mathbf{z} + \mathbf{w} + 10^{-\mathbf{L}}\mathbf{v}$$

where by construction $\mathbf{u}$ and $\mathbf{v}$ satisfy $0 \leq \mathbf{u}, \mathbf{v} \leq 1$. If $\mathbf{x} = \mathbf{y}$ then the displayed equations imply that $10^{-\mathbf{L}} + 10^{-\mathbf{L}}\mathbf{u} = 10^{-\mathbf{L}}\mathbf{v}$. The only way such an equation can hold is if $\mathbf{u} = 0$ and $\mathbf{v} = 1$. The first of these implies that the decimal expansion coefficients for the sum

$$0 = \mathbf{u} = \mathbf{x}_{L+1}10^{-1} + \mathbf{x}_{L+2}10^{-2} + \ldots + \mathbf{x_{L+k}}10^{-\mathbf{k}} + \ldots$$

must satisfy $\mathbf{x}_{\mathbf{k}} = 0$ for all $\mathbf{k} > \mathbf{L}$, and by the lemma the second of these can only happen if the decimal expansion coefficients for the sum

$$\mathbf{1} = \mathbf{v} = \mathbf{y}_{L+1}10^{-1} + \mathbf{y}_{L+2}10^{-2} + \ldots + \mathbf{y}_{L+k}10^{-k} + \ldots$$

satisfy $\mathbf{y}_{\mathbf{k}} = 9$ for all $\mathbf{k} > \mathbf{L}$. Therefore the second alternative holds in Case 2.

Conversely, the standard geometric series argument shows that two numbers with decimal expansions given by the second or third alternatives must be equal. Of course, the two numbers are equal if the first alternative holds, so this completes the proof of the theorem.

One can reformulate the preceding into a strict uniqueness result as follows:

<u>Theorem.</u> *Every positive real number has a unique decimal expansion of the form*

$$\mathbf{a}_{\mathbf{N}}10^{\mathbf{N}} + \mathbf{a}_{\mathbf{N}-1}10^{\mathbf{N}-1} + \ldots + \mathbf{a}_0 + \mathbf{b}_1 10^{-1} + \mathbf{b}_2 10^{-2} + \ldots + \mathbf{b}_{\mathbf{k}}10^{-\mathbf{k}} + \ldots$$

such that $\mathbf{b}_{\mathbf{k}}$ *is nonzero for infinitely many choices of* $\mathbf{k}$.
This follows immediately from the preceding results on different ways of expressing the same real number in decimal form; there is more than one way of writing a number in decimal form if and only if it is an integer plus a finite decimal fraction, and in this case there is only one other way of doing so and all but finitely many digits of the alternate expansion are equal to 9.

EXAMPLE. We can use the preceding result to define real valued functions on an interval in terms of decimal expansions. In particular, if we express an arbitrary real number $\mathbf{x} \in (0, 1]$ as an infinite decimal

$$\mathbf{x} = 0\,.\,\mathbf{b}_1\,\mathbf{b}_2\,\mathbf{b}_3\,\mathbf{b}_4\,\mathbf{b}_5\,\mathbf{b}_6\,\mathbf{b}_7\,\mathbf{b}_8\,\mathbf{b}_9\ldots$$

where infinitely many digits $\mathbf{b}_{\mathbf{k}}$ are nonzero, then we may define a function $\mathbf{f}$ from $(0, 1]$ to itself by the formula

$$\mathbf{f}\,(\mathbf{x}) = 0\,.\,\mathbf{b}_1\,0\,\mathbf{b}_2\,0\,\mathbf{b}_3\,0\,\mathbf{b}_4\,0\,\mathbf{b}_5\,0\,\mathbf{b}_6\,0\,\mathbf{b}_7\,0\,\mathbf{b}_8\,0\,\mathbf{b}_9\,0\ldots$$

and if we extend this function by setting $\mathbf{f}\,(0) = 0$ then we obtain a strictly increasing function on the closed unit interval (verify that the function is strictly increasing!). Note that this function has a jump discontinuity at every finite decimal fraction.

Since every nondecreasing real valued function on a closed interval is Riemann integrable, we know that $\mathbf{f}$ can be integrated. It turns out that the value of this integral is a fairly simple rational number; finding the precise value is left as an exercise for the reader (this is a good illustration of the use of Riemann sums – a natural strategy is to partition the unit interval into pieces whose endpoints are finite decimal fractions with at most $\mathbf{n}$ nonzero terms and to see what happens to the Riemann sums as $\mathbf{n}$ increases).