SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 3

Fall 2005

III. Spaces with special properties

III.1 : Compact spaces – I

Problems from Munkres, § 26, *pp.* 170 – 172

3. Show that a finite union of compact subspaces of X is compact.

SOLUTION.

Suppose that $A_i \subset X$ is compact for $1 \leq i \leq n$, and suppose that $\mathcal{U}$ is a family of open subsets of X whose union contains $\cup_i A_i$. Then for each i there is a finite subfamily $\mathcal{U}_i$ whose union contains A_i. If we take $\mathcal{U}^*$ to be the union of all these subfamilies then it is finite and its union contains $\cup_i A_i$. Therefore the latter is compact.■

7. If Y is compact, show that the projection $\pi_X : X \times Y \to X$ is a closed map.

SOLUTION.

We need to show that if $F \subset X \times Y$ is closed then $\pi_X(F)$ is closed in X, and as usual it is enough to show that the complement is open. Suppose that $x \notin \pi_X(F)$. The latter implies that $\{x\} \times Y$ is contained in the open subset $X \times Y - F$, and by the Tube Lemma one can find an open set $V_x \subset X$ such that $x \in V$ and $V_x \times Y \subset X \times Y - F$. But this means that the open set $V_x \subset X$ lies in the complement of $\pi_X(F)$, and since one has a conclusion of this sort for each such x it follows that the complement is open as required.■

8. Let $f : X \to Y$ be a set-theoretic map of topological spaces, and assume Y is compact Hausdorff. Prove that f is continuous if and only if its graph Γ_f (defined previously) is a closed subset of $X \times Y$.

SOLUTION.

($\Longrightarrow$) We shall show that the complement of the graph is open, and this does not use the compactness condition on Y (although it does use the Hausdorff property). Suppose we are given (x, y) such that $y \neq x$. Then there are disjoint open sets V and W in Y such that $y \in V$ and $f(x) \in W$. Since $f^{-1}(W)$ is open in X and contains x, there is an open set U containing x such that $f(U) \subset W$. It follows that $U \times V$ is an open subset of $X \times Y$ that is disjoint from Γ_f. Since we have such a subset for each point in the complement of Γ_f it follows that $X \times Y - \Gamma_f$ is open and that Γ_f is closed.■

($\Longleftarrow$) As in a previous exercises let $\gamma : X \to \Gamma_f$ be the graph map and let $j : \Gamma_f \to X \times Y$ be inclusion. General considerations imply that the map $\pi_X \circ j$ is continuous and 1–1 onto, and γ is the associated set-theoretic inverse. If we can prove that $\pi_X \circ j$ is a homeomorphism, then γ will also be a homeomorphism and then f will be continuous by a previous exercise. The map in question will be a homeomorphism if it is closed, and it suffices to check that it is a composite of closed mappings. By hypothesis j is the inclusion of a closed subset and therefore j is closed, and the preceding exercise shows that π_X is closed. Therefore the composite is a homeomorphism as claimed and the mapping f is continuous.■

Problem from Munkres, § 27, pp. 177 – 178

2. Let $A \subset X$ be a nonempty subset of a metric space X.

(a) Show that $\mathbf{d}(x, A) = 0$ if and only if $X \in \overline{A}$.

SOLUTION.

The function of x described in the problem is continuous, so its set of zeros is a closed set. This closed set contains A so it also contains $\overline{A}$. On the other hand if $x \notin \overline{A}$ then there is an $\varepsilon > 0$ such that $N_\varepsilon(x) \subset X - \overline{A}$, and in this case it follows that $\mathbf{d}(x, A) \geq \varepsilon > 0$.■

(b) Show that if A is compact then $\mathbf{d}(x, A) = \mathbf{d}(x, a)$ for some $a \in A$.

SOLUTION.

The function $f(a) = \mathbf{d}(x, a)$ is continuous and $\mathbf{d}(x, A)$ is the greatest lower bound for its set of values. Since A is compact, this greatest lower bound is a minimum value that is realized at some point of A.■

(c) Define the ε-*neighborhood* $U(A, \varepsilon)$ to be the set of all u such that $\mathbf{d}(u, A) < \varepsilon$. Show that this is the union of the neighborhoods $N_\varepsilon(a)$ for all $a \in A$.

SOLUTION.

The union is contained in $U(A, \varepsilon)$ because $\mathbf{d}(x, a) < \varepsilon$ implies $\mathbf{d}(x, A) < \varepsilon$. To prove the reverse inclusion suppose that y is a point such that $\delta = \mathbf{d}(y, A) < \varepsilon$. It then follows that there is some point $a \in A$ such that $\mathbf{d}(y, a) < \varepsilon$ because the greater than the greatest lower bound of all possible distances. The reverse inclusion is an immediate consequence of the existence of such a point a.■

(d) Suppose that A is compact and that U is an open set containing A. Prove that there is an $\varepsilon > 0$ such that $A \subset U(A, \varepsilon)$.

SOLUTION.

Let $F = X - U$ and consider the function $g(a) = \mathbf{d}(a, F)$ for $a \in A$. This is a continuous function and it is always positive because $A \cap F = \emptyset$. Therefore it takes a positive minimum value, say ε. If $y \in A \subset U(A, \varepsilon)$ then $\mathbf{d}(a, y) < \varepsilon \leq \mathbf{d}(a, F)$ implies that $y \notin F$, and therefore $U(A, \varepsilon)$ is contained in the complement of F, which is U.■

(e) Show that the preceding conclusion need not hold if A is not compact.

SOLUTION.

Take X to be all real numbers with positive first coordinate, let A be the points of X satisfying $y = 0$, and let U be the set of all points such that $y < 1/|x|$. Then for every $\varepsilon > 0$ there is a point not in U whose distance from A is less than ε. For example, consider the points $(2n, 1/n)$.■

1. Let X be a compact Hausdorff space, and let $f : X \to X$ be continuous. Define $X_1 = X$ and $X_{n+1} = f(X_n)$, and set $A = \bigcap_n X_n$. Prove that A is a nonempty subset of X and $f(A) = A$.

SOLUTION.

Each of the subsets X_n is compact by an inductive argument, and since X is Hausdorff each one is also closed. Since each set in the sequence contains the next one, the intersection of finitely many sets $X_{k(1)}, \cdots, X_{k(n)}$ in the collection is the set $X_{k(m)}$ where $k(m)$ is the maximum of the $k(i)$. Since X is compact, the Finite Intersection property implies that the intersection A of these sets is nonempty. We need to prove that $f(A) = A$. By construction A is the set of all points that lie in the image of the k-fold composite $\circ^k f$ of f with itself. To see that f maps this set into itself note that if $a = [\circ^k f](x_k)$ for each positive integer k then $f(a) = [\circ^k f](\,f(x_k)\,)$ for each k. To see that f maps this set onto itself, note that $a = [\circ^k f](x_k)$ for each positive integer k implies that

$$a = f\left([\circ^k f](x_{k+1})\right)$$

for each k.∎

2. A topological space X is said to be a k-**space** if it satisfies the following condition: *A subset $A \subset X$ is closed if and only if for all compact subsets $K \subset X$, the intersection $A \cap K$ is closed.* It turns out that a large number of the topological spaces one encounters in topology, geometry and analysis are k-spaces (including all metric spaces and compact Hausdorff spaces), and the textbooks by Kelley and Dugundji contain a great deal of information about these k-spaces (another important reference is the following paper by N. E. Steenrod: *A convenient category of topological spaces*, Michigan Mathematical Journal **14** (1967), pp. 133–152).

(a) Prove that if $(X, \mathbf{T})$ is a Hausdorff topological space then there is a unique minimal topology $\mathbf{T}^\kappa$ containing $\mathbf{T}$ such that $\mathcal{K}(X) = (X, \mathbf{T}^\kappa)$ is a Hausdorff k-space.

SOLUTION. In this situation it is convenient to work with topologies in terms of their closed subsets. Let $\mathcal{F}$ be the family of closed subsets of X associated to $\mathbf{T}$ and let $\mathcal{F}^*$ be the set of all subsets E such that $E \cap C$ is closed in X for every compact subset $C \subset X$. If E belongs to $\mathcal{F}$ then $E \cap C$ is always closed in X because C is closed, so $\mathcal{F} \subset \mathcal{F}^*$. We claim that $\mathcal{F}^*$ defines a topology on X and X is a Hausdorff k-space with respect to this topology.

The empty set and X belong to $\mathcal{F}^*$ because they already belong to $\mathcal{F}$. Suppose that E_α belongs to $\mathcal{F}^*$ for all α; we claim that for each compact subset $C \subset X$ the set $C \cap \bigcap_\alpha E_\alpha$ is $\mathcal{F}$-closed in X. This follows because

$$C \;\cap\; \bigcap_\alpha E_\alpha \;\;=\;\; \bigcap_\alpha (C \cap E_\alpha)$$

and all the factors on the right hand side are $\mathcal{F}$-closed (note that they are compact). To conclude the verification that $\mathcal{F}^*$ is a topology, suppose that E_1 and E_2 belong to $\mathcal{F}^*$. Once again let $C \subset X$ be compact, and observe that the set-theoretic equation

$$C \cap (E_1 \cup E_2) \;\;=\;\; (C \cap E_1) \;\cup\; (C \cap E_2)$$

implies the right hand side is $\mathcal{F}$-closed if E_1 and E_2 are.

Therefore $\mathcal{F}^*$ defines the closed subspaces of a topological space; let $\mathbf{T}^\kappa$ be the associated family of open sets. It follows immediately that the latter contains $\mathbf{T}$, and one obtains a Hausdorff space by the following elementary observation: *If $(X, \mathbf{T})$ is a Hausdorff topological space and $\mathbf{T}^*$ is a topology for X containing $\mathbf{T}$, then $(X, \mathbf{T}^*)$ is also Hausdorff.* This is true because the disjoint open sets in $\mathbf{T}$ containing a pair of disjoint points are also (disjoint) open subsets with respect to $\mathbf{T}^*$ containing the same respective points.

We now need to show that $\mathcal{K}(X) = (X, \mathbf{T}^\kappa)$ is a k-space and the topology is the unique minimal one that contains $\mathbf{T}$ and has this property. Once again we switch over to using the closed subsets in all the relevant topologies. The most crucial point is that a subset $D \subset X$ is $\mathcal{F}$-compact if and only if it is $\mathcal{F}^*$-compact; by construction the identity map from $[X, \mathcal{F}^*]$ to $[X, \mathcal{F}]$ is continuous (brackets are used to indicate the subset families are the closed sets), so if D is compact with respect to $\mathcal{F}^*$ it image, which is simply itself, must be compact with respect to $\mathcal{F}$. How do we use this? Suppose we are given a subset $B \subset X$ such that $B \cap D$ is $\mathcal{F}^*$-closed for every $\mathcal{F}^*$-compact subset D. Since the latter is also $\mathcal{F}$-compact and the intersection $B \cap D$ is $\mathcal{F}^*$ compact (it is a closed subspace of a compact space), we also know that $B \cap D$ is $\mathcal{F}$-compact and hence $\mathcal{F}$-closed. Therefore it follows that $\mathcal{F}^*$ is a k-space topology. If we are given the closed subsets for any Hausdorff k-space topology $\mathcal{E}$ containing $\mathcal{F}$, then this topology must contain all the closed sets of $\mathcal{F}^*$. Therefore the latter gives the unique minimal k-space topology containing the topology associated to $\mathcal{F}$.■

(b) Prove that if $f : X \to Y$ is a continuous map of Hausdorff topological spaces, then f is also continuous when viewed as a map from $\mathcal{K}(X) \to \mathcal{K}(Y)$.

SOLUTION.

Suppose that $F \subset Y$ has the property that $F \cap C$ is closed in Y for all compact sets $C \subset Y$. We need to show that $f^{-1}(F) \cap D$ is closed in X for all compact sets $D \subset X$.

If F and D are as above, then $f(D)$ is compact and by the assumption on f we know that

$$f^{-1}\left(F \cap f(D) \right) = f^{-1}(F) \cap f^{-1}\left(f(D)\right)$$

is closed in X with respect to the original topology. Since $f^{-1}\left(f(D)\right)$ contains the closed compact set D we have

$$f^{-1}(F) \cap f^{-1}\left(f(D)\right) \cap D \quad = \quad f^{-1}(F) \cap D$$

and since the left hand side is closed in X the same is true of the right hand side. But this is what we needed to prove.■

3. *Non-Hausdorff topology revisited.* A topological space X is *noetherian* if every nonempty family of open subsets has a maximal element. This class of spaces is also of interest in algebraic geometry.

(a) (*Ascending Chain Condition*) Show that a space X is noetherian if and only if every increasing sequence of open subsets

$$U_1 \subset U_2 \subset \cdots$$

stabilizes; *i.e.*, there is some positive integer N such that $n \geq N$ implies $U_n = U_N$.

SOLUTION.

$(\Longrightarrow)$ The sequence of open subsets has a maximal element; let U_N be this element. Then $n \geq N$ implies $U_N \subset U_n$ by the defining condition on the sequence, but maximality implies the reverse inclusion. Thus $U_N = U_n$ for $n \geq N$.■

$(\Longleftarrow)$ Suppose that the Chain Condition holds but there is a nonempty family $\mathcal{U}$ of open subsets with no maximal element. If we pick any open set U_1 in this family then there is another open set U_2 in the family that properly contains U_2. Similarly, there is another open subset U_3 in the family that properly contains U_2, and we can inductively construct an ascending chain of open subspaces such that each properly contains the preceding ones. This contradicts the Ascending Chain Condition. Therefore our assumption that $\mathcal{U}$ had no maximal element was incorrect.■

(b) Show that a space X is noetherian if and only if every open subset is compact.

SOLUTION.

($\Longrightarrow$) Take an open covering $\{ U_\alpha \}$ of U for which each open subset in the family is nonempty, and let $\mathcal{W}$ be the set of all finite unions of subsets in the open covering. By definition this family has a maximal element, say W. If $W = U$ then U is compact, so suppose W is properly contained in U. Then if $u \in U - W$ and U_0 is an open set from the open covering that contains u, it will follow that the union $W \cap U_0$ is also a finite union of subsets from the open covering and it properly contains the maximal such set W. This is a contradiction, and it arises from our assumption that W was properly contained in U. Therefore U is compact.■

($\Longleftarrow$) We shall show that the Ascending Chain Condition holds. Suppose that we are given an ascending chain

$$U_1 \ \subset \ U_2 \ \subset \ \cdots$$

and let $W = \cup_n U_n$. By our hypothesis this open set is compact so the open covering $\{ U_n \}$ has a finite subcovering consisting of $U_{k(i)}$ for $1 \leq i \leq m$. If we take N to be the maximum of the $k(i)$'s it follows that $W = U_N$ and $U_n = U_N$ for $n \geq N$.■

(c) Show that a noetherian Hausdorff space is finite (with the discrete topology). [*Hint:* Show that every open subset is closed.]

SOLUTION.

We begin by verifying the statement in the hint. If U is open in a noetherian Hausdorff space X, then U is compact and hence U is also closed (since X is Hausdorff). Since U is Hausdorff, one point subsets are closed and their complements are open, so the complements of one point sets are also closed and the one point subsets are also open. Thus a noetherian Hausdorff space is discrete. On the other hand, an infinite discrete space does not satisfy the Ascending Chain Condition (pick an infinite sequence of distinct points x_k and let U_n be the first n points of the sequence. Therefore a noetherian Hausdorff space must also be finite.■

(d) Show that a subspace of a noetherian space is noetherian.

SOLUTION.

Suppose $Y \subset X$ where X is noetherian. Let $\mathcal{V} = \{ V_\alpha \}$ be a nonempty family of open subspaces of Y, write $V_\alpha = U_\alpha \cap Y$ where U_α is open in X, and let $\mathcal{U} = \{ U_\alpha \}$. Since X is noetherian, this family has a maximal element U^*, and the intersection $V^* = U^* \cap Y$ will be a maximal element of $\mathcal{V}$.■

III.2 : Complete metric spaces

Problems from Munkres, § 43, pp. 270 − 271

1. Let X be a metric space.

(a) Suppose that for some $\varepsilon > 0$ that every ε-neighborhood of every point has compact closure. Prove that X is complete.

SOLUTION.

Let $\{ x_n \}$ be a Cauchy sequence in X and choose M so large that $m, n \geq M$ implies $\mathbf{d}(x_m.x_n) < \varepsilon$. Then all of the terms of the Cauchy sequence except perhaps the first $M - 1$

lie in the closure of $N_\varepsilon(x_M)$, which is compact. Therefore it follows that the sequence has a convergent subsequence $\{ x_{n(k)} \}$. Let y be the limit of this subsequence; we need to show that y is the limit of the entire sequence.

Let $\eta > 0$ be arbitrary, and choose $N_1 \geq M$ such that $m, n \geq N_1$ implies $\mathbf{d}(x_m.x_n) < \eta/2$. Similarly, let $N_2 \geq M$ be such that $n(k) > N_2$ implies $\mathbf{d}(x_{n(k)}.y) < \eta/2$. If we take N to be the larger of N_1 and N_2, and application of the Triangle Inequality shows that $n \leq N$ implies $\mathbf{d}(x_n.y) < \eta$. Therefore y is the limit of the given Cauchy sequence and X is complete.■

(b) Suppose that for each $x \in X$ there is an $\varepsilon > 0$ that $N_\varepsilon(x)$ has compact closure. Show by example that X is not necessarily complete.

SOLUTION.

Take $U \subset \mathbf{R}^2$ to be the set of all points such that $xy < 1$. This is the region "inside" the hyperbolas $y = \pm 1/x$ that contains the origin. It is not closed in $\mathbf{R}^2$ and therefore cannot be complete. However, it is open and just like all open subsets U of $\mathbf{R}^2$ if $x \in X$ and $N_\varepsilon(x) \subset U$ then $N_{\varepsilon/2}(x)$ has compact closure in U.

3. Two metrics $\mathbf{d}$ and $\mathbf{e}$ on the same set X are said to be metrically equivalent if the identity maps from $(X, \mathbf{d})$ to $(X, \mathbf{e})$ and vice versa are uniformly continuous.

(b) Show that if $\mathbf{d}$ and $\mathbf{e}$ are metrically equivalent then X is complete with respect to one metric if and only if it is complete with respect to the other.

SOLUTION.

By the symmetry of the problem it is enough to show that if $(X, \mathbf{d})$ is complete then so is $(X, \mathbf{e})$. Suppose that $\{ x_n \}$ is a Cauchy seuence with respect to $\mathbf{e}$; we claim it is also a Cauchy sequence with respect to $\mathbf{d}$. Let $\varepsilon > 0$, and take $\delta > 0$ such that $\mathbf{e}(u, v) < \delta$ implies $\mathbf{d}(u, v) < \varepsilon$. If we choose M so that $m, n \geq M$ implies $\mathbf{e}(x_n, x_m) < \delta$, then we also have $\mathbf{d}(x_n, x_m) < \varepsilon$. Therefore the original Cauchy sequence with respect to $\mathbf{e}$ is also a Cauchy sequence with respect to $\mathbf{d}$. By completeness this sequence has a limit, say y, with respect to $\mathbf{d}$, and by continuity this point is also the limit of the sequence with respect to $\mathbf{e}$.■

6. A space X is said to be *topologically complete* if it is homeomorphic to a complete metric space.

(a) Show that a closed subspace of a topologically complete space is topologically complete.

SOLUTION.

This follows because a closed subspace of a complete metric space is complete.■

(c) Show that an open subspace of a topologically complete space is topologically complete. [*Hint:* If $U \subset X$ and X is complete with respect to $\mathbf{d}$, define a continuous real valued function on U by the formula $\phi(x) = 1/\mathbf{d}(X, X - U)$ and embed U in $X \times \mathbf{R}$ by the function $f(x) = (x, \phi(x))$.]

SOLUTION.

The image of the function f is the graph of ϕ, and general considerations involving graphs of continuous functions show that f maps U homeomorphically onto its image. If this image is closed in $X \times \mathbf{R}$ then by **(a)** we know that U is topologically complete, so we concentrate on proving that $f(U) \subset X \times \mathbf{R}$ is closed. The latter in turn reduces to proving that the image is closed in the complete subspace $\overline{U} \times \mathbf{R}$, and as usual one can prove this by showing that the complement of $f(U)$ is open. The latter in turn reduces to showing that if $(x, t) \in \overline{U} \times \mathbf{R} - f(U)$ then there is an open subset containing (x, t) that is disjoint from U.

Since ϕ is continuous it follows immediately that the graph of ϕ is closed in the open subset $U \times \mathbf{R}$. Thus the open set $U \times \mathbf{R} - f(U)$ is open in $\overline{U} \times \mathbf{R}$, and it only remains to consider points in $\overline{U} - U \times \mathbf{R}$. Let (x,t) be such a point. In this case one has $\mathbf{d}(x, X-U) = 0$. There are three cases depending on whether t is less than, equal to or greater than 0. In the first case we have that $(x,t) \in \overline{U} \times (-\infty, 0)$ which is open in $\overline{U} \times \mathbf{R}$ and contains no points of $f(U)$ because the second coordinates of points in the latter set are always positive. Suppose now that $t = 0$. Then by continuity of distance functions there is an open set $V \subset \overline{U}$ such that $x \in V$ and $\mathbf{d}(y, X-U) < 1$ for all $y \in V$. It follows that $V \times (-1,1)$ contains (x,t) and is disjoint from $f(U)$. Finally, suppose that $t > 0$. Then by continuity there is an open set $V \subset \overline{U}$ such that $x \in V$ and

$$\mathbf{d}(y, X-U) \quad < \quad \frac{2}{3t}$$

for all $y \in V$. It follows that $V \times (t/2, 3t/2)$ contains (x,t) and is disjoint from $f(U)$, and this completes the proof of the final case.■

Additional exercises

1. Show that the Nested Intersection Property for complete metric spaces does not necessarily hold for nested sequences of closed subsets $\{ A_n \}$ if $\lim_{n\to\infty} \operatorname{diam}(A_n) \neq 0$; *i.e.*, in such cases one might have $\cap_n A_n = \emptyset$. [*Hint:* Consider the set A_n of all continuous functions f from $[0,1]$ to itself that are zero on $[\frac{1}{n}, 1]$ and also satisfy $f(0) = 1$.]

SOLUTION.

Follow the hint, and try to see what a function in the intersection would look like. In the first place it has to satisfy $f(0) = 1$, but for each $n > 0$ it must be zero for $t \geq 1/n$. The latter means that the $f(t) = 0$ for all $t > 0$. Thus we have determined the values of f everywhere, but the function we obtained is not continuous at zero. Therefore the intersection is empty. Since every function in the set A_n takes values in the closed unit interval, it follows that if f and g belong to A_n then $\|f - g\| \leq 1$ and thus the diameter of A_n is at most 1 for all n. In fact, the diameter is exactly 1 because $f(0) = 1$.

For the sake of completeness, we should note that each set A_n is nonempty. One can construct a "piecewise linear" function in the set that is zero for $t \geq 1/n$ and decreases linearly from the 1 to 0 as t increases from 0 to $1/n$. (Try to draw a picture of the graph of this function!)■

2. Let ℓ^2 be the set of all real sequences $\mathbf{x} = \{ x_n \}$ such that $\sum_n |x_n|^2$ converges. The results of Exercise 10 on page 128 of Munkres show that ℓ^2 is a normed vector space with the norm

$$|\,\mathbf{x}\,| = \left(\sum_n |x_n|^2 \right)^{1/2} .$$

Prove that ℓ^2 is complete with respect to the associated metric. [*Hint:* If $\mathbf{p}_i$ gives the i^{th} term of an element in ℓ^2, show that $\mathbf{p}_i$ takes Cauchy sequences to Cauchy sequences. This gives a candidate for the limit of a Cauchy sequence in ℓ^2. Show that this limit candidate actually lies in ℓ^2 and that the Cauchy sequence converges to it. See also Royden, *Real Analysis*, Section 6.3, pages 123–127, and also Rudin, *Principles of Mathematical Analysis*, Theorem 11.42 on page 329–330 together with the discussion on the following two pages.]

SOLUTION.

For each positive integer k let $H_k(y)$ be the vector whose first k coordinates are the same as those of y and whose remaining coordinates are zero, and let $T_k = I - H_k$ (informally, these are "head" and "tail" functions). Then H_k and T_k are linear transformations and $|H_k(y)|,\ |T_k(y)| \ \leq |y|$ for all y.

Since Cauchy sequences are bounded there is some $B > 0$ such that $|x_n| \leq B$ for all n.

Let x be given as in the hint. We claim that $x \in \ell^2$; by construction we know that $H_k(x) \in \ell^2$ for all k. By the completeness of $\mathbf{R}^M$ we know that $\lim_{n\to\infty} H_k(x_n) = H_k(x)$ and hence there is an integer P such that $n \geq P$ implies $|H_k(x) - H_k(x_n)| < \varepsilon$. If $n \geq P$ we then have that

$$|H_k(x)| \ \leq \ |H_k(x_n)| + |H_k(x) - H_M(k_n)| \ \leq \ |x_n| + |H_M(x) - H_M(x_n)| \ < \ B + \varepsilon \ .$$

By construction $|x|$ is the least upper bound of the numbers $|H_k(x)|$ if the latter are bounded, and we have just shown the latter are bounded. Therefore $x \in \ell^2$; in fact, the argument can be pushed further to show that $|x| \leq B$, but we shall not need this.

We must now show that x is the limit of the Cauchy sequence. Let $\varepsilon > 0$ and choose M this time so that $n,\, m \geq M$ implies $|x_m - x_n| < \varepsilon/6$. Now choose N so that $k \geq N$ implies $|T_k(x_M)| < \varepsilon/6$ and $|T_k(x)| < \varepsilon/3$; this can be done because the sums of the squares of the coordinates for x_M and x are convergent. If $n \geq M$ then it follows that

$$|T_k(x_n)| \ \leq \ |T_k(x_M)| + |T_k(x_n) - T_k(x_M)| = \ |T_k(x_n)| \ \leq \ |T_k(x_M)| + |T_k(x_n - x_M)| \ \leq$$

$$|T_k(x_M)| + |x_n - x_M| \ \leq \ \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \ = \ \frac{\varepsilon}{3} \ .$$

Choose P so that $P \geq M + N$ and $n \geq P$ implies $|H_N(x) - H_N(x_n)| < \frac{\varepsilon}{3}$. If $n \geq P$ we then have

$$|x - x_n| \ \leq \ |H_N(x) - H_N(x_n)| + |T_N(x) - T_N(x_n)| \ \leq$$

$$|H_N(x) - H_N(x_n)| \ + \ |T_N(x)| \ + \ |T_N(x_n)| \ < \ \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \ = \ \varepsilon \ .$$

This completes the proof.■

III.3 : Implications of completeness

Problems from Munkres, § 48, *pp.* 298 – 300

1. Let X be the countable union $\cup_n\, B_n$. Show that if X is a nonempty Baire space then at least one of the sets $\overline{B_n}$ has a nonempty interior.

SOLUTION.

If the interior of the closure of B_n is empty, then B_n is nowhere dense. Thus if the interior of the closure of each B_n is empty then X is of the first category.■

2. Baire's Theorem implies that $\mathbf{R}$ cannot be written as a countable union of closed subsets having nonempty interiors. Show this fails if the subsets are not required to be closed.

SOLUTION.

View the real numbers as a vector space over the rationals. The previous argument on the existence of bases implies that the set $\{1\}$ is contained in a basis (work out the details!). Let B be such a basis (over the rationals), and let W be the rational subspace spanned by all the remaining vectors in the basis. Then $\mathbf{R}$ is the union of the cosets $c+W$ where c runs over all rational numbers, and this is a countable union. We claim that each of these cosets has a nonempty interior. To see this, note that each coset contains exactly one rational number, while if there interior were nonempty it would contain an open interval and every open interval contains infinitely many rational numbers (*e.g.*, if $c \in (a,b)$ there is a strictly decreasing sequence of rational numbers $r_n \in (c,b)$ whose limit is c).■

4. Show that if every point in X has a neighborhood that is a a Baire space then X is a Baire space. [*Hint:* Use the open set formulation of the Baire conditions in Lemma 48.1 on page 296 of Munkres.]

SOLUTION.

According to the hint, we need to show that if $\{ V_n \}$ is a sequence of open dense subsets in X, then $(\cap_n V_n)$ is dense.

Let $x \in X$, and let W be an open neighborhood of x that is a Baire space. We claim that the open sets $V_n \cap W$ are all dense in W. Suppose that W_0 is a nonempty open subset of W; then W_0 is also open in X, and since $\cap_n V_n$ is dense in X it follows that

$$W_0 \cap (W \cap V_n) = (W_0 \cap W) \cap V_n = W_0 \cap V_n \neq \emptyset$$

for all n. Therefore $V_n \cap W$ is dense in W. Since W is a Baire space and, it follows that

$$\bigcap_n (V_n \cap W) = W \cap \left(\bigcap_n V_n \right)$$

is dense in W.

To show that $(\cap_n V_n)$ is dense in X, let U be a nonempty open subset of X, let $a \in U$, let W_a be an open neighborhood of a that is a Baire space, and let $U_0 = U \cap W$ (hence $a \in U_0$). By the previous paragraph the intersection

$$U_0 \cap \left(\bigcap_n (W \cap V_n) \right)$$

is nonempty, and since this intersection is contained in

$$U \quad \cap \quad \left(\bigcap_n V_n \right)$$

it follows that the latter is also nonempty, which implies that the original intersection $(\cap_n V_n)$ is dense.■

Problem from Munkres, § 27, *pp.* 178 – 179

6. Construct the Cantor set by taking $A_0 = [0,1]$ and defining A_n by removing

$$\bigcup_k \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$$

from A_{n-1}. The intersection $\cap_k A_k$ is the Cantor set C.

PRELIMINARY OBSERVATION. Each of the intervals that is removed from A_{n-1} to construct A_n is entirely contained in the former. One way of doing this is to partition $[0,1]$ into the 3^n intervals

$$\left[\frac{b}{3^n}, \frac{b+1}{3^n}\right]$$

where b is an integer between 0 and $3^n - 1$. If we write down the unique 3-adic expansion of b as a sum

$$\sum_{i=0}^{n-1} a_i \, 3^i$$

where $b_i \in \{0,1,2\}$, then A_k consists of the intervals associated to numbers b such that none of the coefficients b_i is equal to 1. Note that these intervals are pairwise disjoint; if the interval corresponding to b lies in A_k then either the interval corresponding to $b+1$ or $b-1$ is not one of the intervals that are used to construct A_k (the base 3 expansion must end with a 0 or a 2, and thus one of the adjacent numbers has a base 3 expansion ending in a 1). The inductive construction of A_n reflects the fact that for each k the middle third is removed from the closed interval

$$\left[\frac{k}{3^{n-1}}, \frac{k+1}{3^{n-1}}\right] .$$

(a) Show that C is totally disconnected; *i.e.*, the only nonempty connected subspaces are one point subsets.

SOLUTION.

By induction A_k is a union of 2^k pairwise disjoint closed intervals, each of which has length 3^{-k}; at each step one removes the middle third from each of the intervals at the previous step. Suppose that $K \subset C$ is nonempty and connected. Then for each n the set K must lie in one of the 2^n disjoint intervals of length 3^{-n} in A_n. Hence the diameter of K is $\leq 3^{-n}$ for all $n \geq 0$, and this means that the diameter is zero; *i.e.*, K consists of a single point.■

(b) Show that C is compact.

SOLUTION.

By construction C is an intersection of closed subsets o the compact space $[0,1]$, and therefore it is closed and hence compact.■

(c) Show that the endpoints of the 2^n intervals comprising A_n all lie in C.

SOLUTION.

If we know that the left and right endpoints for the intervals comprising A_n are also (respectively) left and right hand endpoints for intervals comprising A_{n+1}, then by induction it will follow that they are similar endpoints for intervals comprising A_{n+k} for all $k \geq 0$ and therefore they will all be points of C. By the descriptions given before, the left hand endpoints for A_n are all numbers of the form $b/3^n$ where b is a nonnegative integer of the form

$$\sum_{i=0}^{n-1} b_i \, 3^i$$

with $b_i = 0$ or 2 for each i, and the right hand endpoints have the form $(b+1)/3^n$ where b has the same form. If b has the indicated form then

$$\frac{b}{3^n} = \frac{3b}{3^{n+1}}, \quad \text{where} \quad 3b = \sum_{i=1}^{n} b_{i-1}\, 3^i$$

shows that $b/3^n$ is also a left hand endpoint for one of the intervals comprising A_{n+1}. Similarly, if $(b+1)/3^n$ is a right hand endpoint for an interval in A_n and b is expanded as before, then the equations

$$\frac{b+1}{3^n} = \frac{3b+3}{3^{n+1}}$$

and

$$(3b+3) - 1 = 3b+2 = 2 + \sum_{i=1}^{n} b_{i-1}\, 3^i$$

show that $(b+1)/3^n$ is also a right hand endpoint for one of the intervals comprising A_{n+1}.■

(d) Show that C has no isolated points.

SOLUTION.

If $x \in C$ then for each n one has a unique closed interval J_n of length 2^{-n} in A_n such that $x \in J_n$. Let $\lambda_n(x)$ denote the left hand endpoint of that interval unless x is that point, and let $\lambda_n(x)$ be the right hand endpoint in that case; we then have $\lambda_n(x) \neq x$. By construction $\left|\lambda_n(x) - x\right| < 2^{-n}$ for all n, and therefore $x = \lim_{n\to\infty} \lambda_n(x)$. On the other hand, by the preceding portion of this problem we know that $\lambda_n(x) \in C$, and therefore we have shown that x is a limit point of C, which means that x is not an isolated point.■

(e) Show that C is uncountable.

SOLUTION.

One way to do this is by using **(d)** and the Baire Category Theorem. By construction C is a compact, hence complete, metric space. It is infinite by **(c)**, and every point is a limit point by **(d)**. Since a countable complete metric space has isolated points, it follows that C cannot be countable.

In fact, one can show that $|C| = 2^{\aleph_0}$. The first step is to note that *if $\{a_k\}$ is an infinite sequence such that $a_k \in \{0,2\}$ for each k, then the series*

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

converges and its sum lies in C.

This assertion may be verified as follows: The infinite series converges by a comparison test with the convergent series such that $a_k = 2$ for all k. Given a point as above, the partial sum

$$\sum_{k=1}^{n} \frac{a_k}{3^k}$$

is a left hand endpoint for one of the intervals comprising A_n. The original point will lie in A^n if the sum of the rest of the terms is $\leq 1/3$. But

$$\sum_{k=n+1}^{\infty} \frac{a_k}{3^k} \leq \sum_{k=n}^{\infty} \frac{2}{3^k} = \frac{2}{3^{n+1}} \cdot \frac{1}{(1-\frac{1}{3})} = \frac{1}{3}$$

so the point does lie in A_n. Since n was artitrary, this means that the sum lies in $\cap_n A_n = C$.

Returning to the original problem of determining $|C|$, we note that the set $\mathcal{A}$ of all sequences described in the assertion is in a natural 1–1 correspondence with the set of all functions from the positive integers to $\{0,1\}$. Let $\mathcal{A}_0$ be the set of all functions whose values are nonzero for infinitely many values of n, and let $\mathcal{A}_1$ be the functions that are equal to zero for all but finitely many values of n. We then have that $|\mathcal{A}_1| = \aleph_0$ and $\mathcal{A}_0$ is infinite (why?). The map sending a function in $\mathcal{A}_0$ to the associated sum of an infinite series is 1–1 (this is just a standard property of base N expansions — work out the details), and therefore we have

$$|\mathcal{A}_0| \;\leq\; |C| \;\leq\; |\mathbf{R}| \;=\; 2^{\aleph_0}$$

and

$$|\mathcal{A}_0| \;=\; |\mathcal{A}_0| + \aleph_0 \;=\; |\mathcal{A}_0| + |\mathcal{A}_1| \;=\; |\mathcal{A}| \;=\; 2^{\aleph_0}$$

which combine to imply $|C| = 2^{\aleph_0}$.■

Additional exercises

1. Let A and B be subspaces of X and Y respectively such that $A \times B$ is nowhere dense in $X \times Y$ (with respect to the product topology). Prove that either A is nowhere dense in X or B is nowhere dense in Y, and give an example to show that "or" cannot be replaced by "and."

SOLUTION.

Suppose that the conclusion is false: *i.e.*, A is not nowhere dense in X and B is not nowhere dense in Y. Then there are nonempty open sets U and V contained in the closures of A and B respectively, and thus we have

$$\emptyset \;\neq\; U \times V \;\subset\; \overline{A} \times \overline{B} \;=\; \overline{A \times B}$$

and therefore $A \times B$ is not nowhere dense in $X \times Y$.

To see that we cannot replace "or" with "and" take $X = Y = \mathbf{R}$ and let A and B be equal to $[0,1]$ and $\{0\}$ respectively. Then A is not nowhere dense in X but $A \times B$ is nowhere dense in $X \times Y$.■

2. Is there an uncountable topological space of the first category?

SOLUTION.

The space $\mathbf{R}^\infty$ has this property because it is the union of the closed nowhere dense subspaces A_n that are defined by the condition $x_i = 0$ for $i > n$.■

3. Let X be a metric space. A map $f : X \to X$ is said to be an *expanding similarity* of X if f is onto and there is a constant $C > 1$ such that

$$\mathbf{d}\,(\,f(u),\,f(v)\,) \;=\; C \cdot \mathbf{d}(u,v)$$

for all u, $v \in X$ (hence f is 1–1 and uniformly continuous). Prove that every expanding similarity of a complete metric space has a unique fixed point. [*Hint and comment:* Why does f have an inverse that is uniformly continuous, and why does $f(x) = x$ hold if and only if $f^{-1}(x) = x$? If $X = \mathbf{R}^n$ and a similarity is given by $f(x) = cAx + b$ where A comes from an orthogonal matrix and either $0 < c < 1$ or $c > 1$, then one can prove the existence of a unique fixed point directly using linear algebra.]

SOLUTION.

First of all, the map f is 1–1 onto; we are given that it is onto, and it is 1–1 because $u \neq v$ implies $\mathbf{d}(\,f(u),\,f(v)\,) > \mathbf{d}(u,v) > 0$. Therefore f has an inverse, at least set-theoretically, and we denote f^{-1} by T.

We claim that T satisfies the hypotheses of the Contraction Lemma. The proof of this begins with the relations

$$\mathbf{d}(\,T(u),\,T(v)\,) = \mathbf{d}(\,f^{-1}(u),\,f^{-1}(v)\,) = \frac{1}{C1}\mathbf{d}\left(\,f\left(f^{-1}(u)\right),\,f\left(f^{-1}(v)\right)\,\right) = \mathbf{d}(u,v)\ .$$

Since $C > 1$ it follows that $0 < 1/C < 1$ and consequently the hypotheses of the Contraction Lemma apply to our example.

Therefore T has a unique fixed point p; we claim it is also a fixed point for f. We shall follow the hint. Since T is 1–1 and onto, it follows that $x = T(T^{-1}(x))$ and that $T(x) = x \implies x = T^{-1}(x)$; the converse is even easier to establish, for if $x = T^{-1}(x)$ the application of T yields $T(x) = x$. Since there is a unique fixed point p such that $T(p) = p$, it follows that there is a unique point, in fact the same one as before, such that $p = T^{-1}(p)$, which is equal to $f(p)$ by definition.■

THE CLASSICAL EUCLIDEAN CASE.

This has two parts. The first is that every expanding similarity of $\mathbf{R}^n$ is expressible as a so-called affine transformation $T(v) = cAv + b$ where A is given by an orthogonal matrix. The second part is to verify that each transformation of the type described has a unique fixed point. By the formula, the equation $T(x) = x$ is equivalent to the equation $x = cAx + b$, which in turn is equivalent to $(I - cA)x = b$. The assertion that T has a unique fixed point is equivalent to the assertion that this linear equation has a unique solution. The latter will happen if $I - cA$ is invertible, or equivalently if $\det(I - cA) \neq 0$, and this is equivalent to saying that c^{-1} is not an eigenvalue of A. But if A is orthogonal this means that $|Av| = |v|$ for all v and hence the only possible eigenvalues are ± 1; on the other hand, by construction we have $0 < c^{-1} < 1$ and therefore all of the desired conclusions follow. The same argument works if $0 < c < 1$, the only change being that one must substitute $c^{-1} > 1$ for $0 < c^{-1} < 1$ in the preceding sentence.■

4. Consider the sequence $\{x_n\}$ defined recursively by the formula

$$x_{n+1} \quad = \quad \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)\ .$$

If $x_0 > 0$ and $x_0^2 > a$, show that the sequence $\{\,x_n\,\}$ converges by proving that

$$\varphi(x) \quad = \quad \frac{1}{2}\left(x + \frac{a}{x}\right)$$

is a contraction mapping on $[\sqrt{a}, x_0]$.

SOLUTION.

We need to show that φ maps $[\sqrt{a}, x_0]$ into itself and that the absolute value of its derivative takes a maximum value that is less than 1.

In this example, the best starting point is the computation of the derivative, which is simply an exercise in first year calculus:

$$\varphi'(x) \quad = \quad \frac{1}{2}\left(1 - \frac{a}{x^2}\right)$$

This expression is an increasing function of x over the set $[\sqrt{a}, +\infty)$; its value at $\sqrt{a}$ is 0 and the limit at $+\infty$ is $1/2$. In particular, the absolute value of the derivative on $[\sqrt{a}, x_0]$ is less than $1/2$, and by the Mean Value Theorem the latter in turn implies that φ maps the interval in question to

$$\left[\sqrt{a}, \frac{\sqrt{a}+x_0}{2}\right]$$

which is contained in $[\sqrt{a}, x_0]$.■

III.4 : Connected spaces

Problems from Munkres, § 23, p. 152

3. Let A be a connected subspace of X, and let $\{A_\alpha\}$ be a collection of connected subspaces of X whose union is B. Show that if $A \cap A_\alpha \neq \emptyset$ for each α then $A \cup B$ is connected.

SOLUTION.

Suppose that C is a nonempty open and closed subset of $Y = A \cup B$. Then by connectedness either $A \subset C$ or $A \cap C = \emptyset$; without loss of generality we may assume the first holds, for the argument in the second case will follow by interchanging the roles of C and $Y - C$. We need to prove that $C = Y$.

Since each A_α for each α we either have $A_\alpha \subset C$ or $A_\alpha \cap C = \emptyset$. In each case the latter cannot hold because $A \cap A_\alpha$ is a nonempty subset of C, and therefore A_α is contained in C for all α. Therefore $C = Y$ and hence Y is connected.■

4. Show that if X is an infinite set then it is connected in the finite complement topology.

SOLUTION.

Suppose that we can write $X = C \cup D$ where C and D are disjoint nonempty open and closed subsets. Since C is open it follows that either D is finite or all of X; the latter cannot happen because $X - D = C$ is nonempty. On the other hand, since C is closed it follows that either C is finite or $C = X$. Once again, the latter cannot happen because $X - C = D$ is nonempty. Thus X is a union of two finite sets and must be finite, which contradicts our assumption that X is infinite. This forces X to be connected.■

5. A space is *totally disconnected* if its connected components are all one point subsets. Show that a discrete space is totally disconnected. Does the converse hold?

SOLUTION.

Suppose that C is a maximal connected subset; then C also has the discrete topology, and the only discrete spaces that are connected are those with at most one point.

There are many examples of totally disconnected spaces that are not discrete. The set of rational numbers and all of its subspaces are fundamental examples; here is the proof: Let x and y be distinct rational numbers with $x < y$. Then there is an irrational number r between them, and the identity

$$\mathbf{Q} = \Big(\mathbf{Q} \cap (-\infty, r)\Big) \quad \cup \quad \Big(\mathbf{Q} \cap (r, +\infty, r)\Big)$$

gives a separation of $\mathbf{Q}$, thus showing that x and y lie in different connected components of Q. But x and y were arbitrary so this means that no pair of distinct points can lie in the same connected component of Q. The argument for subsets of $\mathbf{Q}$ proceeds similarly.■

9. Let A and B be proper subsets of the connected spaces X and Y. Prove that $X \times Y - A \times B$ is connected.

SOLUTION.

A good way to approach this problem is to begin by drawing a picture in which $X \times Y$ is a square and $A \times B$ is a smaller concentric square. It might be helpful to work with this picture while reading the argument given here.

Let $x_0 \in X - A$ and $y_0 \in Y - B$, and let C be the connected component of (x_0, y_0) in $X \times Y - A \times B$. We need to show that C is the entire space, and in order to do this it is enough to show that given any other point (x, y) in the space there is a connected subset of $X \times Y - A \times B$ containing it and (x_0, y_0). There are three cases depending upon whether or not $x \in A$ or $y \in B$ (there are three options rather than four because we know that both cannot be true).

If $x \notin A$ and $y \notin B$ then the sets $X \times \{y_0\}$ and $\{x\} \times Y$ are connected subsets such that (x_0, y_0) and (x, y_0) lie in the first subset while (x, y_0) and (x, y) lie in the second. Therefore there is a connected subset containing (x, y) and (x_0, y_0) by Exercise 3.— Now suppose that $x \in A$ but $y \notin B$. Then the two points in question are both contained in the connected subset $X \times \{y\} \cup times\{x_0\} \times Y$. Finally, if $x \notin A$ but $y \in B$, then the two points in question are both contained in the connected subset $X \times \{y_0\} \cup \{x\} \times Y$. Therefore the set $X \times Y - A \times B$ is connected.■

12. [*Assuming Y is a closed subset of X.*] Let Y be a nonempty [*closed*] subset of X, and assume that A and B form a separation of $X - Y$. Prove that $Y \cup A$ and $Y \cup B$ are connected. [Munkres does not explicitly assume $Y \neq \emptyset$, but without this assumption the conclusion is false.]

SOLUTION.

Since $X - Y$ is open in X and A and B are disjoint open subsets of $X - Y$, it follows that A and B are open in X. The latter in turn implies that $X \cup A$ and $Y \cup B$ are both closed in X.

We shall only give the argument for $X \cup A$; the proof for $X \cup B$ is the similar, the only change being that the roles of A and B are interchanged. Once again, it might be helpful to draw a picture.

Suppose that C is a nonempty proper subset of $Y \cup A$ that is both open and closed, and let $D = (Y \cup A) - C$. One of the subsets C, D must contain some point of Y, and without loss of generality we may assume it is C. Since Y is connected it follows that all of Y must be contained in C. Suppose that $D \neq \emptyset$. Since D is closed in $Y \cup A$ and the latter is closed in X, it follows that D is closed in X. On the other hand, since D is open in $Y \cup A$ and disjoint from Y it follows that D is open in A, which is the complement of Y in $Y \cup A$. But A is open in X and therefore D is open in X. By connectedness we must have $D = X$, contradicting our previous observation that $D \cap Y = \emptyset$. This forces the conclusion that D must be empty and hence $Y \cup A$ is connected.■

Problems from Munkres, § 24, *pp.* 157 – 159

1. **(b)** Suppose that there exist embeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an example that X and Y need not be homeomorphic.

SOLUTION.

Take $X = (0, 1) \cup (2, 3)$ and $Y = (0, 3)$, let f be inclusion, and let g be multiplication by $1/3$. There are many other examples. In the spirit of part **(a)** of this exercise, one can also take $X = (0, 1)$, $Y = (0, 1]$, f = inclusion and g = multiplication by $1/2$, and similarly one can take $X = (0, 1]$, $Y = [0, 1]$, f = inclusion and $g(t) = (t + 1)/2$.■

FOOTNOTE.

Notwithstanding the sort of examples described in the exercises, a result of S. Banach provides a "resolution" of X and Y if there are continuous embeddings $f : X \to Y$ and $g : Y \to X$; namely, there are decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $X_1 \cap X_2 = Y_1 \cap Y_1 = \emptyset$ and there are homeomorphisms $X_1 \to X_2$ and $Y_1 \to Y_2$. The reference is S. Banach, *Un théorème sur les transformations biunivoques*, Fundamenta Mathematicæ **6** (1924), 236–239.

Additional exercises

1. Prove that a topological space X is connected if and only if every open covering $\mathcal{U} = \{U_\alpha\}$ has the following property: For each pair of sets U, V in $\mathcal{U}$ there is a sequence of open sets $\{U_0, U_1, \cdots U_n\}$ in $\mathcal{U}$ such that $U = U_0$, $V = U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for all i.

SOLUTION.

Define a binary relation $\sim$ on X such that $u \sim v$ if and only if there are open subsets U and V in $\mathcal{U}$ such that $u \in U$, $v \in V$, and there is a sequence of open sets $\{U_0, U_1, \cdots U_n\}$ in $\mathcal{U}$ such that $U = U_0$, $V = U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for all i. This is an equivalence relation (verify this in detail!). Since every point lies in an open subset that belongs to $\mathcal{U}$ it follows that the equivalence classes are open. Therefore the union of all but one equivalence class is also open, and hence a single equivalence classs is also closed in the space. If X is connected, this can only happen if there is exactly one equivalence class.■

2. Let X be a connected space, and let $\mathcal{R}$ be an equivalence relation that is locally constant (for each point x all points in some neighborhood of x lie in the $\mathcal{R}$-equivalence class of x). Prove that $\mathcal{R}$ has exactly one equivalence class.

SOLUTION.

This is similar to some arguments in the course notes and to the preceding exercise. The hypothesis that the relation is locally constant implies that equivalence classes are open. Therefore the union of all but one equivalence class is also open, and hence a single equivalence classs is also closed in the space. If X is connected, this can only happen if there is exactly one equivalence class.■

3. Prove that an open subset in $\mathbf{R}^n$ can have at most countably many components. Give an example to show this is not necessarily true for closed sets.

SOLUTION.

The important point is that $\mathbf{Q}^n$ is dense in $\mathbf{R}^n$; given a point $x \in \mathbf{R}^n$ with coordinates x_i and $\varepsilon > 0$ we can choose rational numbers q_i such that $|q_i - x_i| < \varepsilon/n$; if y has coordinates q_i then $|x - y| < \varepsilon$ follows immediately. Since $\mathbf{R}^n$ is locally connected, the components of open sets are open, and therefore we conclude that every component of a nonempty open set $U \subset \mathbf{R}^n$ contains some point of $\mathbf{Q}^n$. Picking one such point for each component we obtain a 1–1 map from the set of components into the countable set $\mathbf{Q}^n$.

The Cantor set is an example of a closed subset of $\mathbf{R}$ for which the components are the one point subsets and there are uncountably many points.■

III.5 : Variants of connectedness

Problem from Munkres, § 24, pp. 157 – 159

8. **(b)** If $A \subset X$ and A is path connected, is $\overline{A}$ path connected?

SOLUTION.

Not necessarily. Consider the graph of $\sin(1/x)$ for $x > 0$. It closure is obtained by adding the points in $\{0\} \times [-1, 1]$ and we have shown that the space consisting of the graph and this closed segment is not path connected.■

Problem from Munkres, § 25, pp. 162 – 163

10. [*Only* **(a)**, **(b)** *and examples A and B from* **(c)**.] Let X be a space, and define a binary relation by $x \sim y$ if there is no separation $X = A \cup B$ into disjoint open subsets such that $x \in A$ and $y \in B$.

(a) Show that this is an equivalence relation; the equivalence classes are called *quasicomponents*.

SOLUTION.

The proofs that $\sim$ is reflexive and symmetric are very elementary and left to the reader. Regarding transitivity, suppose that $x \sim y$ and $y \sim z$ and that we are given a separation of X as $A \cup B$. Without loss of generality we may assume that $x \in A$ (otherwise interchange the roles of A and B in the argument). Since $x \sim y$ it follows that $y \in A$, and the latter combines with $y \sim z$ to show that $z \in A$. Therefore if we are given a separation of X as above, then x and z lie in the same piece, and since this happens for all separations it follows that $x \sim z$.■

(b) Show that each [connected] component of X is contained in a quasicomponent and that the components and quasicomponents are the same if X is locally connected.

SOLUTION.

Let C be a connected component of X. Then $A \cap C$ is an open and closed subset of C and therefore it is either empty or all of C. In the first case $C \subset B$ and in the second case $C \cap B$. In either case it follows that all points of C lie in the same equivalence class. – If X is locally connected then each connected component is both closed and open. Therefore, if x and y lie in different components, say C_x and C_y, then $X = C_x \cup X - C_x$ defines a separation such that x lies in the first subset and y lies in the second, so that $x \sim y$ is false if x and y do not lie in the same connected component. Combining this with the previous part of the exercise, if X is locally connected then $x \sim y$ if and only if x and y lie in the same connected component.■

(c) Let K denote the set of all reciprocals of positive integers, and let $-K$ denote the set $(-1)\cdot K =$ all negatives of points in K. Determine the components, path components and quasicomponents of the following subspaces of $\mathbf{R}^2$:

$$A = (K \times [0,1]) \cup \{(0,0)\} \cup \{(0,1)\}$$

$$B = A \cup ([0,1] \times \{0\})$$

[*Note:* Example C in this part of the problem was not assigned; some comments appear below.]

SOLUTION.

Trying to prove this without any pictures would probably be difficult at best and hopelessly impossible at worst.

Case A. We claim that the connected components are the closed segments $\{1/n\} \times [0,1]$ and the one point subsets $\{(0,0)\}$ and $\{(0,1)\}$. — Each segment is connected and compact (hence closed), and we claim each is also an open subset of A. This follows because the segment $\{1/n\}\times[0,1]$ is the intersection of A with the open subset

$$\left(\frac{1}{2}\left[\frac{1}{n}+\frac{1}{n+1}\right]\ ,\ \frac{1}{2}\left[\frac{1}{n}+\frac{1}{n-1}\right]\right) \times \mathbf{R}\ .$$

Since the segment $\{1/n\} \times [0,1]$ is open, closed and connected, it follows that it must be both a component and a quasicomponent. It is also an arc component because it is arcwise connected. This leaves us with the two points $(0,0)$ and $(0,1)$. They cannot belong to the same component because they do not form a connected set. Therefore each belongs to a separate component and also to a separate path component. The only remaining question iw whether or not they determine the same quasicomponent. To show that they do lie in the same quasicomponent, it suffices to check that if $A = U \cup V$ is a separation of X into disjoint open subsets then both points lie in the same open set. Without loss of generality we may as well assume that the origin lies in U. It then follows that for all n sufficiently large the points $(1/n, 0)$ all lie in U, and the latter implies that all of the connected segments $\{1/n\} \times [0,1]$ is also lie in U for n sufficiently large. Since $(0,1)$ is a limit point of the union of all these segments (how?), it follows that $(0,1)$ also lies in U. This implies that $(0,1)$ lies in the same quasicomponent of A as $(0,0)$.■

Case B. This set turns out to be connected but not arcwise connected. We claim that the path components are given by $\{(0,1)\}$ and its complement. Here is the proof that the complement is path connected: Let P be the path component containing all points of $[0,1] \times \{0\}$. Since the latter has a nontrivial intersection with each vertical closed segment $\{1/n\} \times [0,1]$ it follows that all of these segments are also contained in P, and hence P consists of all points of A except perhaps $(0,1)$. Since $(0,1)$ is a limit point of P (as before) it follows that B is connected and thus there is only one component and one quasicomponent. We claim that there are two path components. Suppose the extra point $(0,1)$ also lies in P. Then, for example, there will be a continuous curve $\alpha : [0,1] \to A$ joining $(0,1)$ to $(1/2,1)$. Let t_0 be the maximum point in the subset of $[0,1]$ where the first coordinate is zero. Since the first coordinate of $\alpha(a)$ is 1, we must have $t_0 < 1$. Since $A \cap \{0\} \times \mathbf{R}$ is equal to $\{(0,0)\} \cup \{(0,1)\}$, it follows that α is constant on $[0, t_0]$, and by continuity there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies that the *second* coordinate of $\alpha(t)$ is greater than $1/2$. If $t \in (t_0, t_0 + (\delta/2))$, then the first coordinate of $\alpha(t)$ is positive and the second is greater than $1/2$. In fact we can choose some t_1 in the open interval such that the first coordinate is irrational. But there are no points in $B \cap (\frac{1}{2}, +\infty)$ whose first coordinates are irrational, so we have a contradiction. The latter arises because we assumed the existence of a continuous curve joining $(0,1)$ to another point in B. Therefore no such curve can exist and (0.1) does not belong to the path component P. Hence B has two path components, and one of them contains only one point.■

Case C. This was not assigned, but we note that this space is connected and each of the (infinitely many) closed segments given in the definition is a separate path component.■

Additional exercises

1. Prove that a compact locally connected space has only finitely many components.

SOLUTION.

In a locally connected space the connected components are open (and pairwise disjoint). These sets form an open covering and by compactness there is a finite subcovering. Since no proper subcollection of the set of components is an open covering, this implies that the set of components must be finite.■

2. Give an example to show that if X and Y are locally connected metric spaces and $f : X \to Y$ is continuous then $f(X)$ is not necessarily locally connected.

SOLUTION.

Let $Y = \mathbf{R}^2$ and let $X \subset \mathbf{R}^2$ be the union of the horizontal half-line $(0, \infty) \times \{0\}$ and the vertical closed segment $\{-1\} \times [-1, 1]$. These subsets of X are closed in X and pairwise disjoint. Let $f : X \to \mathbf{R}^2$ be the continuous map defined on $(0, \infty) \times \{0\}$ by the formula $f(t, 0) = (t, \sin(1/t))$ and on $\{-1\} \times [-1, 1]$ by the formula $f(-1, s) = (0, s)$. The image $f(X)$ is then the example of a non-locally connected space that is described in the course notes.■